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### Publication Date

2019

Peer reviewed|Thesis/dissertation

Advances in the Theory of Linear Dynamical Systems Through Coordinate Decoupling

by

Rubens Goncalves Salsa Junior

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Engineering - Mechanical Engineering

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Fai Ma, Chair  
Professor Oliver O'Reilly  
Professor Per-Olof Persson

Summer 2019

## Abstract

Advances in the Theory of Linear Dynamical Systems Through Coordinate Decoupling

by

Rubens Goncalves Salsa Junior

Doctor of Philosophy in Engineering - Mechanical Engineering

University of California, Berkeley

Professor Fai Ma, Chair

Coordinate coupling in linear dynamical systems is a known barrier to analysis and design. Using recent developments in the theory of decoupling, three problems on the theory of linear systems are tackled. These independent problems have the common characteristic that partial solutions documented in the open literature require explicit, or implicit, coordinate decoupling. The first problem studied is that of converting the equations of motion of multi-degree-of-freedom (MDOF) systems into a form devoid of the velocity term. In this connection, it is shown that MDOF systems can always be converted by an invertible transformation into a canonical form specified by two diagonal coefficient matrices associated with the generalized acceleration and displacement. As an important by-product, a damped linear system that possesses three symmetric and positive definite coefficients can always be recast as an undamped and decoupled system. Secondly, the characterization of the free motion of MDOF damped systems is undertaken. Using the methodology of phase synchronization, it is shown that the free response of a MDOF passive system can be completely characterized by its spectrum. Furthermore, damping ratio for MDOF damped systems can be constructed as a direct extension of the damping ratio for SDOF systems and it can be used to predict oscillatory behavior. Lastly, a comprehensive study is reported on the inverse problem of linear Lagrangian dynamics, which is concerned with finding a scalar function, termed Lagrangian, such that the associated Euler-Lagrange equations are equivalent to the assigned equations of motion. Contrary to popular beliefs, it is shown that many coupled linear systems do not admit Lagrangian functions. In addition, Lagrangian functions generally cannot be determined by system decoupling, but a scalar function that plays the role of a Lagrangian function can be determined for any linear system by decoupling. This generalized Lagrangian function produces the equations of motion and it contains information on system properties, yet it satisfies a modified version of the EulerLagrange equations. A necessary and sufficient condition for generalized Lagrangian functions to be equivalent to Lagrangian functions is also derived.

To Luiza and Capitu

*“Dez na maneira e no tom  
Você é o cheiro bom  
Da madeira do meu violão  
Você é a festa da Penha,  
A feira de São Cristovão,  
É a Pedra do Sal  
Você é a Intrépida Trupe  
A Lona de Guadalupe  
Você é o Leme e o Pontal  
  
Nunca me deixa na mão  
Você é a canção que consigo  
Escrever afinal  
Você é o Buraco Quente  
A Casa da Mãe Joana  
É a Vila Isabel,  
Você é o Largo do Estácio,  
Curva de Copacabana  
Tudo que o Rio me deu  
  
Pé do meu samba  
Chão do meu terreiro  
Mão do meu carinho  
Glória em meu Outeiro  
Tudo para o coração  
De um brasileiro”*

— Pé do Meu Samba, Caetano Veloso

# Contents

<b>List of Select Symbols</b>	<b>iv</b>
<b>List of Figures</b>	<b>vi</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Objectives . . . . .	2
1.2 Organization . . . . .	4
<b>2 A Review of the Theory of Decoupling Equations of Motion</b>	<b>5</b>
2.1 Modal Analysis . . . . .	5
2.2 Phase Synchronization . . . . .	6
2.2.1 Methodology for Decoupling Non-defective Systems . . . . .	8
2.2.1.1 Relation to Classical Modal Analysis . . . . .	11
2.2.1.2 Illustrative Example: a Diagonalizable System . . . . .	13
2.2.2 Methodology for Decoupling Defective Systems . . . . .	15
2.2.2.1 A Possible Simplification . . . . .	17
2.2.2.2 Illustrative Example: a Defective System . . . . .	19
2.2.3 Phase Synchronization as Isospectral Transformation . . . . .	20
2.2.3.1 Illustrative Example: Isospectral Decoupling . . . . .	21
<b>3 A Canonical Form of the Equation of Motion</b>	<b>24</b>
3.1 Problem Formulation . . . . .	24
3.1.1 SDOF Systems . . . . .	25
3.1.2 Classically Damped Systems . . . . .	25
3.2 Generation of the canonical form . . . . .	26
3.2.1 Distinct Eigenvalues . . . . .	26
3.2.1.1 Uniqueness of the Canonical Form . . . . .	27
3.2.1.2 Illustrative Example: a Non-symmetric System . . . . .	28
3.2.2 Defective Systems . . . . .	31
3.2.2.1 Illustrative Example: a Defective Passive System . . . . .	31
3.3 Conclusions . . . . .	33

<b>4</b>	<b>Characterization of Free Motion of Passive Systems</b>	<b>34</b>
4.1	Characterization of Oscillatory Behavior by Eigenvalues . . . . .	35
4.1.1	Background on SDOF Systems . . . . .	35
4.1.2	MDOF Systems . . . . .	36
4.1.2.1	Illustrative Example: Characterization by Eigenvalues . . .	37
4.2	Characterization of Oscillatory Behavior by Damping Ratio . . . . .	40
4.2.1	Damping Ratio for Systems and Oscillatory Behavior . . . . .	41
4.2.1.1	Illustrative Example: Prediction of Oscillatory Behavior . .	44
4.2.1.2	Illustrative Example: Upper Bound . . . . .	45
4.2.1.3	Illustrative Example: Lower Bound . . . . .	45
4.2.2	Invariance Under Equivalence Transformation . . . . .	46
4.2.3	Damping Ratio Exceeding One . . . . .	47
4.2.3.1	Illustrative Example: Underdamped Degrees of Freedom . .	47
4.2.3.2	Illustrative Example: Critically Damped Degree of Freedom	47
4.2.4	Damping Ratio Equal to One . . . . .	48
4.2.4.1	Illustrative Example: Underdamped and Overdamped De- grees of Freedom . . . . .	48
4.3	Conclusions . . . . .	48
<b>5</b>	<b>The Inverse Problem of Lagrangian Dynamics</b>	<b>50</b>
5.1	Problem Statement . . . . .	51
5.1.1	Lagrangians for SDOF Systems . . . . .	51
5.1.2	Lagrangians for Classically Damped Systems . . . . .	52
5.2	Generalized Lagrangian Functions for Systems Possessing Distinct Eigenvalues	53
5.2.1	Transformation of Euler-Lagrange Equations . . . . .	54
5.2.1.1	Illustrative Example: a Symmetric System . . . . .	55
5.2.1.2	Illustrative Example: a Gyroscopic System . . . . .	56
5.2.2	Relating Lagrangians for Coupled and Decoupled Systems . . . . .	57
5.2.2.1	Illustrative Example: Simultaneously Diagonalizable Coeffi- cients . . . . .	60
5.3	Generalized Lagrangian Functions for Defective Systems . . . . .	61
5.3.1	Illustrative Example: a Defective System . . . . .	61
5.4	Existence of Lagrangians for Linear Systems . . . . .	62
5.4.1	Deductions Using Helmholtz Conditions . . . . .	64
5.4.2	Illustrative Example: Construction of Lagrangian Function . . . . .	65
5.5	Conclusions . . . . .	65
<b>6</b>	<b>Conclusion</b>	<b>67</b>
	<b>Bibliography</b>	<b>69</b>

# List of Select Symbols

$n$	number of degrees of freedom
$t$	time
$\mathbf{q}$	response vector of the coupled system
$\mathbf{M}$	acceleration coefficient of the coupled system
$\mathbf{C}$	velocity coefficient of the coupled system
$\mathbf{K}$	displacement coefficient of the coupled system
$\mathbf{f}(t)$	forcing vector of the coupled system
$\mathbf{p}$	response vector of the decoupled system
$\mathbf{D}$	velocity coefficient of the decoupled system
$\mathbf{\Omega}$	displacement coefficient of the decoupled system
$\mathbf{g}(t)$	forcing vector of the decoupled system
$\lambda_j$	$j$ th eigenvalue
$\mathbf{u}_j$	$j$ th eigenvector of the undamped system
$\mathbf{U}$	modal matrix of the undamped system
$\mathbf{v}_j$	$j$ th eigenvector of the quadratic eigenvalue problem
$m_j$	algebraic multiplicity of $j$ th eigenvalue
$n_c$	number of complex conjugate pairs of eigenvalues
$n_r$	number of pairs of real eigenvalues from the quadratic eigenvalues problem
$\mathbf{\Lambda}_1$	diagonal matrix of primary eigenvalues of the quadratic eigenvalue problem
$\mathbf{V}_1$	matrix of primary eigenvectors of the quadratic eigenvalue problem
$\mathbf{\Lambda}_2$	diagonal matrix of secondary eigenvalues of the quadratic eigenvalue problem
$\mathbf{V}_2$	matrix of secondary eigenvectors of the quadratic eigenvalue problem
$\mathbf{I}$	identity matrix or order $n$
$\mathbf{T}_j$	$j$ th submatrix of state space decoupling transformation matrix
$\mathbf{G}_j$	$j$ th transformation matrix for forcing vector
$\mathbf{S}_j$	$j$ th submatrix of state space inverse decoupling transformation matrix
$\rho_j$	geometric multiplicity of the $j$ th eigenvalue
$\mathbf{v}_s^j$	$s$ th eigenvector or generalized eigenvector of the $j$ th repeated eigenvalue

$\mathbf{J}_p$	Jordan matrix of the decoupled system
$\mathbf{J}_q$	Jordan matrix of the coupled system
$\mathbf{V}_p$	matrix of pairing vectors for the decoupled system
$\mathbf{V}_q$	matrix of eigenvectors and generalized eigenvectors of the coupled system
$\mathbf{0}$	zero matrix or zero vector
$\mathbf{x}$	response vector of system in canonical form
$\mathbf{B}$	matrix coefficient of displacement of system in canonical form
$\mathbf{h}(t)$	forcing vector of system in canonical form
$N_r$	different ways to pair the real eigensolutions
$i$	imaginary unit ( $i = \sqrt{-1}$ )
$\zeta$	damping ratio
$\zeta_{pj}$	damping ration of $j$ th degree of freedom
$L(\mathbf{q}, \dot{\mathbf{q}}, t)$	Lagrangian function for the coupled system
$L(\mathbf{p}, \dot{\mathbf{p}}, t)$	Lagrangian function for the decoupled system
$\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$	generalized Lagrangian function
$d_j$	$j$ th diagonal entry of $\mathbf{D}$
$b_j$	$j$ th diagonal entry of $\mathbf{\Omega}$



# List of Figures

1	Flowchart for decoupling a second-order linear system by modal analysis. . . . .	7
2	Indexing of eigenvalues, with all eigenvalues assumed to be either negative or with negative real part. This is true for passive systems [61]. . . . .	9
3	Flowchart for decoupling a second-order linear system by phase synchronization. . . . .	12
4	The response $\mathbf{q}(t)$ of the non-symmetric coupled system is compared to the response $\mathbf{p}(t)$ of the decoupled system. Phase synchronization decouples (1.1) into real SDOF systems (1.3). . . . .	14
5	The response $\mathbf{q}(t)$ of the coupled defective system and the response $\mathbf{p}(t)$ of the decoupled system. . . . .	20
6	The response $\mathbf{q}(t)$ of the coupled defective system is compared with the response of the decoupled systems obtained by isospectral decoupling and eigenvalue preserving decoupling. The initial conditions used for the simulation are $\mathbf{q}(0) = [1 \quad -1]^T$ and $\dot{\mathbf{q}}(0) = [-2 \quad -1]^T$ . . . . .	23
7	The excitation $\mathbf{h}(t)$ of the canonical system (3.19), the excitation $\mathbf{g}(t)$ of the decoupled system with coefficients (3.17) and the excitation $\mathbf{f}(t)$ of the original system (3.14) are shown. . . . .	29
8	The responses of the canonical system (3.19), the decoupled system with coefficients (3.17) and the original system (3.14) are shown. . . . .	30
9	The response $\mathbf{q}(t)$ of the coupled defective system and the response $\mathbf{p}(t)$ of the decoupled system: complex conjugate eigenvalues originate an underdamped degree of freedom with $\zeta_{p1} = 0.1878$ , distinct real eigenvalues originate an overdamped degree of freedom with $\zeta_{p2} = 2.4269$ and repeated real eigenvalues originate a critically damped degree of freedom with $\zeta_{p3} = 1$ . The initial conditions used for the simulation are $\mathbf{q}(0) = [1 \quad 2 \quad 3]^T$ and $\dot{\mathbf{q}}(0) = \mathbf{0}$ . . . . .	39
10	Response $\mathbf{q}(t)$ of system (1.1) with coefficients (4.24) and the response $\mathbf{p}(t)$ of the decoupled system obtained by phase synchronization. The initial conditions $\mathbf{q}(0) = [1 \quad -1]^T$ and $\dot{\mathbf{q}}(0) = \mathbf{0}$ were used in the simulation. Even though each degree of freedom is overdamped, the overdamping condition is not satisfied. . . . .	42

- 11 Response  $\mathbf{q}(t)$  of system (1.1) with coefficients (4.40) and the response  $\mathbf{p}(t)$  of the decoupled system obtained by phase synchronization. The initial conditions  $\mathbf{q}(0) = [1 \ -1]^T$  and  $\dot{\mathbf{q}}(0) = \mathbf{0}$  were used in the simulation. Since  $\zeta < 1$ , at least one degree of freedom is underdamped and the response  $\mathbf{q}(t)$  has oscillatory behavior. . . . . 45

## Acknowledgments

The completion of my doctoral dissertation would not have been possible without the support of many people along my journey at U.C. Berkeley. I consider myself extremely lucky to be advised by Professor Fai Ma. His unwavering support through my years in Berkeley was fundamental so I could achieve my goals, and his scholarly advice led me to paths I never thought possible. Working with him has been the highest point of my academic life. I also want to thank Professor Daniel T. Kawano. Our many exchanges and collaboration during his year-long visit made my graduate student life much richer. I would also like to take a moment to thank Professor O'Reilly and Professor Casey. I learned so much from being their GSI and from our discussions during office hours and lectures. Their dedication on teaching was truly inspiring; they both served as role models for the teacher I wish to one day become.

I must also thank CAPES and the Brazilian government for supporting four years of my graduate studies at Cal through the Science without Borders fellowship. The support made me eager to contribute to the advancement of science in Brazil. I must also thank the Mechanical Engineering department and the graduate division for also supporting me through part of my studies.

I am specially grateful for my parents, Catarina and Rubens. They had none to little formal education, but they always supported my academic endeavours. Last, but not least, I am extremely thankful for the support my wife Luiza gave me through these five years. She always believed in me; all I did was for her. Finally, to our daughter Capitu, our love: we are extremely happy to have had the time to see you grow. It was lovely to study and finish my dissertation with you on my lap.

# Chapter 1

## Introduction

The equation of motion of linear systems is one of the most commonly used equations in science and engineering. In its typical form, the equation of motion of a  $n$ -degree-of-freedom linear dynamical system has the vector-matrix representation

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t), \quad (1.1)$$

where all quantities are real and dots represent differentiation with respect to time  $t \geq 0$ . The generalized coordinate

$$\mathbf{q}(t) = [q_1 \quad q_2 \quad \dots \quad q_n]^T \quad (1.2)$$

and the excitation  $\mathbf{f}(t)$  are  $n$ -dimensional column vectors. The initial conditions  $\mathbf{q}(0)$  and  $\dot{\mathbf{q}}(0)$  as well as  $\mathbf{f}(t)$  are known. The constant coefficients  $\mathbf{M}$ ,  $\mathbf{C}$ , and  $\mathbf{K}$  are arbitrary matrices of order  $n \times n$ , and thus (1.1) represents the so-called linear non-conservative systems [1]. However,  $\mathbf{M}$  is assumed nonsingular here. This assumption is not unduly restrictive, as it might be possible to initially decrease the number of degrees of freedom to ensure that  $\mathbf{M}$  is nonsingular [2]. Equation of motion (1.1) constitutes a set of  $n$  ordinary differential equations that are mutually dependent since the coefficient matrices are most commonly not diagonal. System (1.1) is said to be *coupled* in this case.

It is important to notice that the coefficient matrices allow the system concerned to be classified. For example, an undamped gyroscopic system possesses a skew-symmetric coefficient of velocity  $\mathbf{C}$  [3]. If a system is elastic and non-circulatory, then the coefficient of displacement  $\mathbf{K}$  is symmetric. And so one may go on. Of particular significance is the class of non-gyroscopic, non-circulatory, passive systems characterized by three symmetric and positive definite matrices, in which case  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$  are termed mass, damping and stiffness matrices, respectively. For brevity, this class of systems is referred to as passive or as damped linear systems. There should be no denying that the bulk of existing literature on linear vibration and structural dynamics deals implicitly or explicitly with passive systems [2–16].

The properties of single-degree-of-freedom (SDOF) systems, a realization of equation (1.1) with  $n = 1$ , are well understood [2–16], but coordinate coupling in multi-degree-of-freedom (MDOF) systems is viewed as a barrier to analysis and design in both theoretical and

practical viewpoints. It is common for engineers to seek a methodology that transforms (1.1) into a set of  $n$  independent ordinary differential equations so as to treat problems in linear system dynamics in a more tractable manner [17–30]. Such a process is termed *decoupling*: a methodology that transforms (1.1) into a set of  $n$  independent ordinary differential equations in the variable  $\mathbf{p}$ :

$$\ddot{\mathbf{p}}(t) + \mathbf{D}\dot{\mathbf{p}}(t) + \mathbf{\Omega}\mathbf{p}(t) = \mathbf{g}(t), \quad (1.3)$$

for which the coefficient matrices  $\mathbf{D}$  and  $\mathbf{\Omega}$  are real and diagonal.

In *The Theory of Sound* in 1894, Lord Rayleigh [17] already expounded on the significance of system decoupling to solve problems in engineering and introduced the concept of proportional damping. Under the assumption of classical damping, for which proportional damping is just a special case, a passive system can be decoupled by a congruence transformation [31, 32] in the  $n$ -dimensional configuration space, a process that is the time-honored method of modal analysis [18]. There is no particular reason why a linear system should be classically damped. Thus, a damped system cannot in general be decoupled by modal analysis. Indeed, experimental modal testing suggests that no physical system is strictly classically damped [33].

For non-classically damped systems, efforts may be direct to approximately diagonalize  $\mathbf{C}$ . One common procedure is to replace the normalized damping matrix by a selected diagonal matrix. This method of decoupling the system minimizes the error bound for diagonally-dominant matrices [20–22]. To avoid this issue, researchers [34, 35] extended classical modal analysis to a process of complex modal analysis in the state space to treat non-classically damped systems. However, the state space approach has never appealed to practicing engineers because the dimension of the state space is twice the number of degrees of freedom and, mainly, there is little physical insight associated with different elements of complex modal analysis, whereas classical modal analysis is amenable to physical interpretation. For example, each normal mode represents a physical profile of vibration [2, 3, 5–18]. Therefore efforts in engineering are concentrated on decoupling (1.1) in the real  $n$ -dimensional configuration space (1.3).

Ma and Caughey showed that no time-invariant linear transformation can be used to decouple all damped systems in the configuration space [19]. Therefore any universal decoupling transformation in the configuration space must be at least time-varying or even nonlinear. Recently, Ma et al. [23–27] developed the method of *phase synchronization*, which decouples (1.1) into (1.3) through a real, nonlinear, and time-varying transformation, a methodology devoid of the classical damping assumption.

## 1.1 Objectives

Using earlier developments [23–27] on the decoupling of the equations of motion, the objective is to use the method of phase synchronization as the main theoretical tool to investigate three problems in the dynamics of linear systems that have once been impeded by coordinate coupling, i.e., problems that have required explicitly, or implicitly, coordinate

decoupling, such as modal analysis and its underlying assumptions (e.g., classical damping). The problems concerned are summarized in the following three questions.

**Can equations of motion be transformed so as to eliminate the coefficient of velocity?** It is well known that the equation of motion of a SDOF passive system can be converted into an undamped system by an invertible transformation [5, 36–38]. This undamped form is sometimes referred to as the normal form of a SDOF system. If a MDOF passive system is classically damped, then it can be decoupled by modal analysis into a series of independent SDOF systems. The  $n$  independent SDOF equations can be cast into normal form, and the original coupled system is then transformed into a form devoid of the velocity term [28]. The key to successful reduction of the equation of motion of classically damped linear systems, as described, is decoupling in real space.

In this connection, an invertible transformation that converts linear systems of the form (1.1) into a canonical form specified by two diagonal coefficient matrices associated with the generalized acceleration and displacement is sought. As an important by-product, a damped linear system that possesses three symmetric and positive definite coefficients can then always be recast as an undamped and decoupled system.

**How can one characterize free motion of passive systems?** A passive or damped  $n$ -degree-of-freedom linear system is characterized by three symmetric and positive definite coefficient matrices. In many engineering applications, it is important to determine the effect of damping on the overall response, i.e., to find out whether the damped system exhibits oscillatory or nonoscillatory behaviors. For example, in engineering design applications one needs to know how oscillations can be suppressed by varying certain system parameters; a very slow decay of the oscillations is often desirable in electrical networks [39].

Characterization of the free motion of a SDOF damped system (equation (1.1) with  $n = 1$ , for which the three coefficients are simply positive real numbers) is well understood [2–16]: the nature of damped free motion can be determined by inspection of the viscous damping ratio, which is a scalar defined by the coefficients of the differential equation of motion. In vibration terminology, this damping ratio shows whether the system is underdamped, critically damped or overdamped. Oscillatory behavior can be observed in underdamped systems, while the free response of an overdamped system decays exponentially without oscillations. Critical damping represents the boundary between oscillatory and non-oscillatory behaviors.

While a similar criteria for determining oscillatory behavior of free motion of MDOF damped systems is desired, the situation here is less clear. Various criteria for determining the free response characteristics of MDOF damped systems have been reported in the literature. Some proposed solutions [40–44] rely upon the assumption of classical damping. The purpose is to study the free response characteristics of a MDOF damped system using phase synchronization, making it possible to mirror known results for SDOF damped systems.

**Can decoupling be used to solve the inverse problem in Lagrangian dynamics?** The direct problem of Lagrangian dynamics involves the derivation of equations of motion of a system with an assigned Lagrangian function. In contrast, the inverse problem is concerned with finding a scalar function such that the associated Euler-Lagrange equations are

equivalent to the assigned equations of motion. This scalar function, termed a Lagrangian, provides a highly compact form of storage of information on system properties; it generates the equations of motion among other things. Owing to utility in several fields, the inverse problem, sometimes referred to as the inverse problem of the calculus of variations, is a well-trodden problem that has attracted the attention of many researchers in the past century [29, 30, 45–59].

A general solution to the inverse problem has never been reported in the open literature. However, Lagrangian functions have already been determined for SDOF systems [45–53]. Solution of the inverse problem for MDOF systems poses a greater challenge because the equations of motion are usually coupled. Using solutions for SDOF systems, Udwadia [29] obtained Lagrangians for classically damped linear systems through modal analysis. A comprehensive study is needed for the evaluation of Lagrangian functions for linear systems possessing symmetric or non-symmetric coefficient matrices.

## 1.2 Organization

To answer these questions, this dissertation is organized as follows: the canonical form of linear systems is addressed in chapter 3, the problem of characterizing the free vibration of damped systems is presented in chapter 4 while the inverse problem of Lagrangian dynamics is tackled in chapter 5. A summary of findings is provided in chapter 6. Chapter 2 offers background material on phase synchronization, which is extensively used throughout this dissertation, and sets up the notation. Several examples are provided for illustration throughout the text. While each problem is tackled by coordinate decoupling, Chapters 3, 4 and 5 may be read independently and are self-contained, only referring to results in chapter 2 for theoretical background. Some of the materials in this dissertation were drawn upon co-authored journal publications in [59] and [60].

## Chapter 2

# A Review of the Theory of Decoupling Equations of Motion

The purpose of this chapter is to provide an overview of the fundamental aspects of the theory of linear system decoupling and set up notation used throughout the dissertation. The organization of this chapter is as follows. In section 2.1, modal analysis is reviewed to place decoupling of general linear systems into perspective. The main aspects of phase synchronization are reported in section 2.2.

### 2.1 Modal Analysis

Suppose the coefficient matrices in (1.1) are symmetric and positive definite. Associated with this passive system is the symmetric eigenvalue problem

$$\mathbf{K}\mathbf{u} = \lambda\mathbf{M}\mathbf{u}. \quad (2.1)$$

Owing to the positive definiteness of  $\mathbf{M}$  and  $\mathbf{K}$ , all eigenvalues  $\lambda_j$  ( $j = 1, 2, \dots, n$ ) are real and positive, and the corresponding eigenvectors  $\mathbf{u}_j$  are real and orthogonal with respect to either  $\mathbf{M}$  or  $\mathbf{K}$  [32]. Denote the  $n \times n$  modal and spectral matrices, respectively, by

$$\mathbf{U} = [\mathbf{u}_1 \quad \dots \quad \mathbf{u}_n], \quad \mathbf{\Omega} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}. \quad (2.2)$$

Upon normalization of the eigenvectors with respect to the mass matrix, the generalized orthogonality of the eigenvectors can be expressed as

$$\mathbf{U}^T \mathbf{M} \mathbf{U} = \mathbf{I}, \quad \mathbf{U}^T \mathbf{K} \mathbf{U} = \mathbf{\Omega}. \quad (2.3)$$

Using the modal transformation  $\mathbf{q} = \mathbf{U}\mathbf{p}$ , where  $\mathbf{p}$  is an  $n$ -dimensional vector of principal coordinates, a passive system represented by equation (1.1) can be converted into the form



(1.3) with

$$\mathbf{g}(t) = \mathbf{U}^T \mathbf{f}(t). \quad (2.4)$$

The symmetric matrix

$$\mathbf{D} = \mathbf{U}^T \mathbf{C} \mathbf{U} \quad (2.5)$$

is referred to as the modal damping matrix. Note that the mass matrix  $\mathbf{M}$  and the stiffness matrix  $\mathbf{K}$  have been diagonalized by the modal transformation, which means an undamped system can always be decoupled by modal analysis. Any coupling in a linear system occurs ultimately through damping.

A system is classically damped if it can be decoupled by classical modal analysis, whereby the modal damping matrix  $\mathbf{D}$  in (2.5) is diagonal. In Section 97 of *The Theory of Sound* in 1894, Lord Rayleigh [17] provided a sufficient requirement, referred to as proportional damping, under which a system is classically damped:

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K} \quad (2.6)$$

for some scalar constants  $\alpha$  and  $\beta$ . In 1965, Caughey and O'Kelly [18] established that a necessary and sufficient condition under which a system is classically damped is

$$\mathbf{C} \mathbf{M}^{-1} \mathbf{K} = \mathbf{K} \mathbf{M}^{-1} \mathbf{C}. \quad (2.7)$$

Figure 1 provides the algorithm for modal analysis.

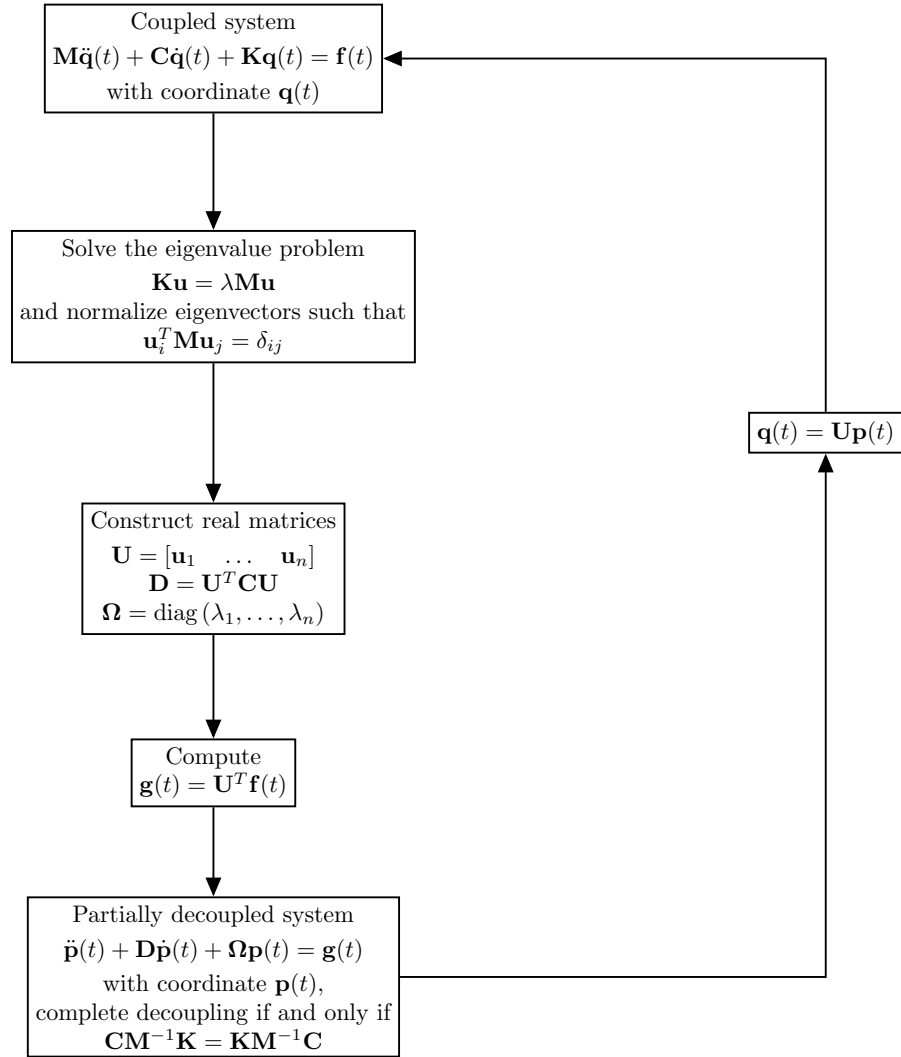
There is, of course, no particular reason why condition (2.7) should be satisfied. Indeed, experimental modal testing suggests that no physical system is strictly classically damped [33]. In general, a linear dynamical system is non-classically damped and it cannot be decoupled by classical modal analysis.

## 2.2 Phase Synchronization

Recently, modal analysis has been extended such that linear systems can be decoupled in real space [23–27]. Specifically, a real and invertible transformation has been determined to convert equation (1.1) into (1.3) for which the  $n \times n$  coefficient matrices  $\mathbf{D}$  and  $\mathbf{\Omega}$  are real and diagonal. Unless (1.1) represents a classically damped passive system,  $\mathbf{D}$  and  $\mathbf{\Omega}$  are not the same as the modal damping and spectral matrices, respectively. There are no scientific restrictions on this extension of modal analysis, which is termed the method of phase synchronization. All parameters required for decoupling are obtained through the solution of the quadratic eigenvalue problem

$$(\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K}) \mathbf{v} = \mathbf{0}. \quad (2.8)$$

Solution of (2.8) yields  $2n$  eigenvalues  $\lambda_j$  ( $j = 1, 2, \dots, 2n$ ) [61–64], where the set of eigenvalues  $\{\lambda_1, \lambda_2, \dots, \lambda_{2n}\}$  is termed the spectrum of (1.1). If  $\mathbf{M}$  is invertible, then all eigenvalues are finite. An eigenvalue might have several partial multiplicities. The number



**Fig. 1** Flowchart for decoupling a second-order linear system by modal analysis.

of occurrences of an eigenvalue is its algebraic multiplicity, which is the sum of its partial multiplicities. The number of partial multiplicities is the geometric multiplicity. An eigenvalue is simple if it occurs only once; such an eigenvalue has unit partial, algebraic, and geometric multiplicities. A repeated eigenvalue is semisimple when its algebraic and geometric multiplicities coincide. Associated with a semisimple eigenvalue  $\lambda_j$  with algebraic multiplicity  $m_j \leq n$  are  $m_j$  eigenvectors  $\mathbf{v}_s^j \neq \mathbf{0}$  ( $s = 1, 2, \dots, m_j$ ) that are the linearly independent column vectors in the null space of  $(\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K})$ . Consequently, a simple eigenvalue  $\lambda$  has a single eigenvector  $\mathbf{v} \neq \mathbf{0}$  that is the solution of (2.8). When an eigenvalue  $\lambda_j$  of equation (2.8) occurs  $m_j$  times and a full complement of  $m_j$  independent eigenvectors cannot be found, system (1.1) is defective. Thus, a system represented by (1.1) is said to be non-defective when every repeated eigenvalue of (2.8) possesses a full complement of independent eigenvectors.

To streamline the presentation, in section 2.2.1 it is assumed that all eigenvalues of (2.8) are distinct, which guarantees that the system concerned is non-defective. Relaxation of this assumption to include defective systems, which must possess repeated eigenvalues, will be considered in section 2.2.2. Perhaps an alternative viewpoint on repeated eigenvalues should be brought up. If  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$  are randomly chosen from a uniform distribution, the probability that all eigenvalues of (2.8) are distinct is one [23]. In this sense, almost all linear systems are characterized by distinct eigenvalues.

### 2.2.1 Methodology for Decoupling Non-defective Systems

To provide a concise exposition, an implementation of phase synchronization to decouple non-defective systems with distinct eigenvalues is summarized as a series of tasks. The theory of phase synchronization is expounded in [23–27], and formulas provided in [25] are drawn upon in this presentation.

**Task 1. Solve the quadratic eigenvalue problem (2.8) and index the eigensolutions.**

There are  $2n$  eigensolutions, and any complex eigensolutions occur in complex conjugate pairs. Suppose  $2n_c$  eigenvalues are complex and the remaining  $2n_r = 2(n - n_c)$  are real. The  $n_c$  complex eigenvalues with positive imaginary parts are arranged in order of increasing magnitude of their imaginary parts as the first  $n_c$  eigenvalues such that

$$S_1 = \{\lambda_1, \lambda_2, \dots, \lambda_{n_c} : 0 < \text{Im}[\lambda_1] \leq \text{Im}[\lambda_2] \leq \dots \leq \text{Im}[\lambda_{n_c}]\}. \quad (2.9)$$

Since  $\text{Im}[\lambda_j]$  can often be regarded as a frequency of vibration, this is consistent with the convention of arranging frequencies in order of increasing magnitude. Enumerate the remaining  $n_c$  complex eigenvalues, which are the complex conjugates with negative imaginary parts, in such a way that

$$S_3 = \{\lambda_{n+1} = \bar{\lambda}_1, \lambda_{n+2} = \bar{\lambda}_2, \dots, \lambda_{n+n_c} = \bar{\lambda}_{n_c}\}, \quad (2.10)$$

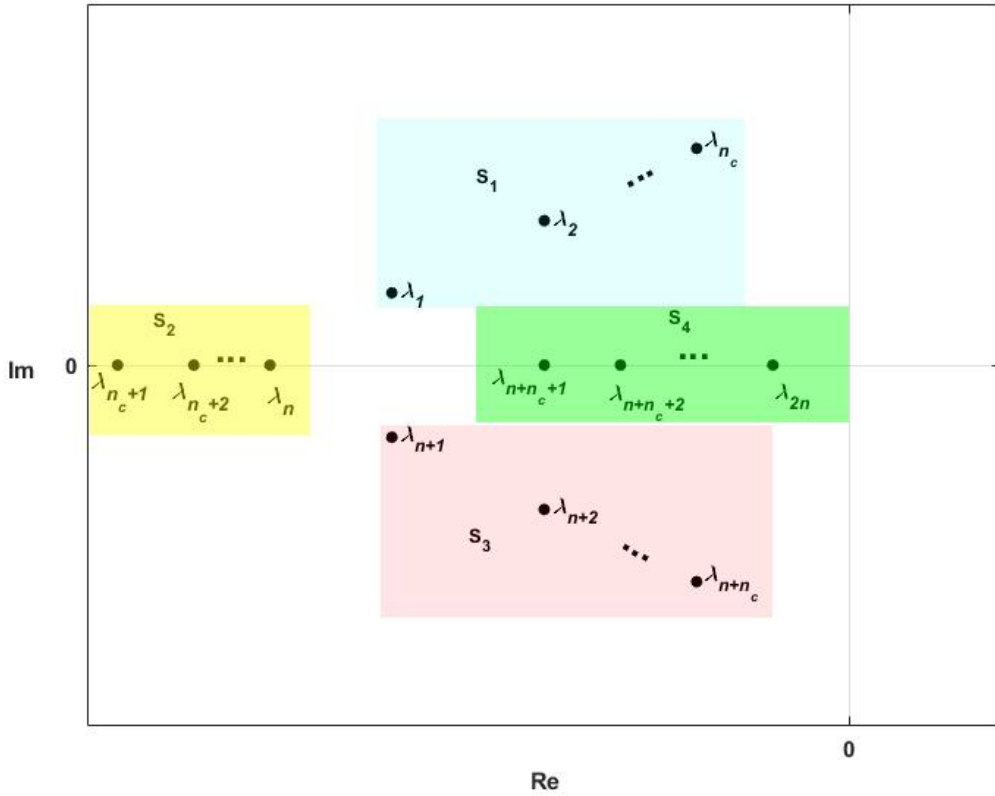
where  $\bar{\lambda}_1$  denotes the complex conjugate of  $\lambda_1$ , and so on. Thus,  $S_1 \cup S_3$  contains the entire set of  $2n_c$  complex eigenvalues. The real eigenvalues are arranged in accordance with a primary-secondary pairing scheme [24]. Among the  $2n_r$  real eigenvalues, the  $n_r$  largest eigenvalues are referred to as primary eigenvalues and the  $n_r$  smallest eigenvalues are termed secondary eigenvalues. Enumerate the  $n_r$  real secondary eigenvalues in order of increasing magnitude such that

$$S_2 = \{\lambda_{n_c+1}, \lambda_{n_c+2}, \dots, \lambda_n : \lambda_{n_c+1} < \lambda_{n_c+2} < \dots < \lambda_n\}. \quad (2.11)$$

Enumerate the remaining  $n_r$  real primary eigenvalues also in order of increasing magnitude so that

$$S_4 = \{\lambda_{n+n_c+1}, \lambda_{n+n_c+2}, \dots, \lambda_{2n} : \lambda_{n+n_c+1} < \lambda_{n+n_c+2} < \dots < \lambda_{2n}\}. \quad (2.12)$$

Thus,  $S_2 \cup S_4$  contains the entire set of  $2n_r$  real eigenvalues under the constraint that  $\sup(S_2) < \inf(S_4)$ . The  $2n$  eigenvalues are partitioned into four disjoint subsets. A different indexing scheme for the eigensolutions may be used, subject to the requirement that complex conjugate eigensolutions are always paired. Figure 2 provides a visual aid for the indexing of eigenvalues.



**Fig. 2** Indexing of eigenvalues, with all eigenvalues assumed to be either negative or with negative real part. This is true for passive systems [61].

**Task 2. Normalize the eigenvectors of equation (2.8).**

After the eigensolutions have been indexed, the  $2n$  eigenvectors are normalized in accordance with

$$2\lambda_j \mathbf{v}_j^T \mathbf{v}_j + \mathbf{v}_j^T \mathbf{C} \mathbf{v}_j = \lambda_j - \lambda_{n+j} \quad (2.13)$$

and

$$2\lambda_{n+j} \mathbf{v}_{n+j}^T \mathbf{v}_{n+j} + \mathbf{v}_{n+j}^T \mathbf{C} \mathbf{v}_{n+j} = \lambda_{n+j} - \lambda_j \quad (2.14)$$

for  $1 \leq j \leq n$ . The above normalization reduces to mass-normalization for undamped or classically damped passive systems [2, 33]. This task is optional, and a different scheme for normalizing the eigenvectors may also be used.

**Task 3. Construct the decoupled form (1.3) using the eigenvalues and eigenvectors of (2.8).**

Using the indexed eigensolutions, assemble the following  $n \times n$  matrices:

$$\mathbf{\Lambda}_1 = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad \mathbf{\Lambda}_2 = \begin{bmatrix} \lambda_{n+1} & & \\ & \ddots & \\ & & \lambda_{2n} \end{bmatrix} \quad (2.15)$$

and

$$\mathbf{V}_1 = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n], \quad \mathbf{V}_2 = [\mathbf{v}_{n+1} \quad \mathbf{v}_{n+2} \quad \dots \quad \mathbf{v}_{2n}]. \quad (2.16)$$

The real and diagonal coefficients of equation (1.3) are given by

$$\mathbf{D} = -(\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2), \quad \mathbf{\Omega} = \mathbf{\Lambda}_1 \mathbf{\Lambda}_2. \quad (2.17)$$

The excitation  $\mathbf{g}(t)$  of (1.3) is given in terms of  $\mathbf{f}(t)$  by

$$\mathbf{g}(t) = \left( \mathbf{D} + \mathbf{I} \frac{d}{dt} \right) \mathbf{G}_1 \mathbf{f}(t) + \mathbf{G}_2 \mathbf{f}(t), \quad (2.18)$$

where  $\mathbf{G}_1$  and  $\mathbf{G}_2$  are real  $n \times n$  matrices computed in accordance with

$$\mathbf{G}_1 = \left[ (\mathbf{V}_1 \mathbf{\Lambda}_1 - \mathbf{V}_2 \mathbf{\Lambda}_2 \mathbf{V}_2^{-1} \mathbf{V}_1)^{-1} + (\mathbf{V}_2 \mathbf{\Lambda}_2 - \mathbf{V}_1 \mathbf{\Lambda}_1 \mathbf{V}_1^{-1} \mathbf{V}_2)^{-1} \right] \mathbf{M}^{-1} \quad (2.19)$$

and

$$\mathbf{G}_2 = \left[ \mathbf{\Lambda}_1 (\mathbf{V}_1 \mathbf{\Lambda}_1 - \mathbf{V}_2 \mathbf{\Lambda}_2 \mathbf{V}_2^{-1} \mathbf{V}_1)^{-1} + \mathbf{\Lambda}_2 (\mathbf{V}_2 \mathbf{\Lambda}_2 - \mathbf{V}_1 \mathbf{\Lambda}_1 \mathbf{V}_1^{-1} \mathbf{V}_2)^{-1} \right] \mathbf{M}^{-1}. \quad (2.20)$$

**Task 4. Construct the real decoupling transformations in the configuration and state spaces.**

Assemble the following real  $n \times n$  matrices:

$$\mathbf{T}_1 = (\mathbf{V}_1 \mathbf{\Lambda}_2 - \mathbf{V}_2 \mathbf{\Lambda}_1) (\mathbf{\Lambda}_2 - \mathbf{\Lambda}_1)^{-1}, \quad (2.21)$$

$$\mathbf{T}_2 = (\mathbf{V}_2 - \mathbf{V}_1) (\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_1)^{-1}, \quad (2.22)$$

$$\mathbf{T}_3 = (\mathbf{V}_1 - \mathbf{V}_2) (\boldsymbol{\Lambda}_1 \boldsymbol{\Lambda}_2) (\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_1)^{-1}, \quad (2.23)$$

$$\mathbf{T}_4 = (\mathbf{V}_2 \boldsymbol{\Lambda}_2 - \mathbf{V}_1 \boldsymbol{\Lambda}_1) (\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_1)^{-1}. \quad (2.24)$$

The configuration space decoupling transformation can be expressed as

$$\mathbf{q}(t) = \left( \mathbf{T}_1 + \mathbf{T}_2 \frac{d}{dt} \right) \mathbf{p}(t) - \mathbf{T}_2 \mathbf{G}_1 \mathbf{f}(t). \quad (2.25)$$

When cast in the state space, the decoupling transformation takes the form

$$\begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \mathbf{S} \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_1 \mathbf{f}(t) \end{bmatrix}, \quad (2.26)$$

where the  $2n \times 2n$  real and invertible matrix  $\mathbf{S}$  is given by

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_3 & \mathbf{S}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_3 & \mathbf{T}_4 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \boldsymbol{\Lambda}_1 & \boldsymbol{\Lambda}_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 \\ \mathbf{V}_1 \boldsymbol{\Lambda}_1 & \mathbf{V}_2 \boldsymbol{\Lambda}_2 \end{bmatrix}^{-1}. \quad (2.27)$$

By expansion, the  $n \times n$  submatrices  $\mathbf{S}_j$  ( $j = 1, 2, 3, 4$ ) have the representations

$$\mathbf{S}_1 = [(\mathbf{V}_1 \boldsymbol{\Lambda}_1)^{-1} - (\mathbf{V}_2 \boldsymbol{\Lambda}_2)^{-1}] [\mathbf{V}_1 (\mathbf{V}_1 \boldsymbol{\Lambda}_1)^{-1} - \mathbf{V}_2 (\mathbf{V}_2 \boldsymbol{\Lambda}_2)^{-1}]^{-1}, \quad (2.28)$$

$$\mathbf{S}_2 = (\mathbf{V}_1^{-1} - \mathbf{V}_2^{-1}) [(\mathbf{V}_1 \boldsymbol{\Lambda}_1) \mathbf{V}_1^{-1} - (\mathbf{V}_2 \boldsymbol{\Lambda}_2) \mathbf{V}_2^{-1}]^{-1}, \quad (2.29)$$

$$\mathbf{S}_3 = (\mathbf{V}_1^{-1} - \mathbf{V}_2^{-1}) [\mathbf{V}_1 (\mathbf{V}_1 \boldsymbol{\Lambda}_1)^{-1} - \mathbf{V}_2 (\mathbf{V}_2 \boldsymbol{\Lambda}_2)^{-1}]^{-1}, \quad (2.30)$$

$$\mathbf{S}_4 = (\boldsymbol{\Lambda}_1 \mathbf{V}_1^{-1} - \boldsymbol{\Lambda}_2 \mathbf{V}_2^{-1}) [(\mathbf{V}_1 \boldsymbol{\Lambda}_1) \mathbf{V}_1^{-1} - (\mathbf{V}_2 \boldsymbol{\Lambda}_2) \mathbf{V}_2^{-1}]^{-1}. \quad (2.31)$$

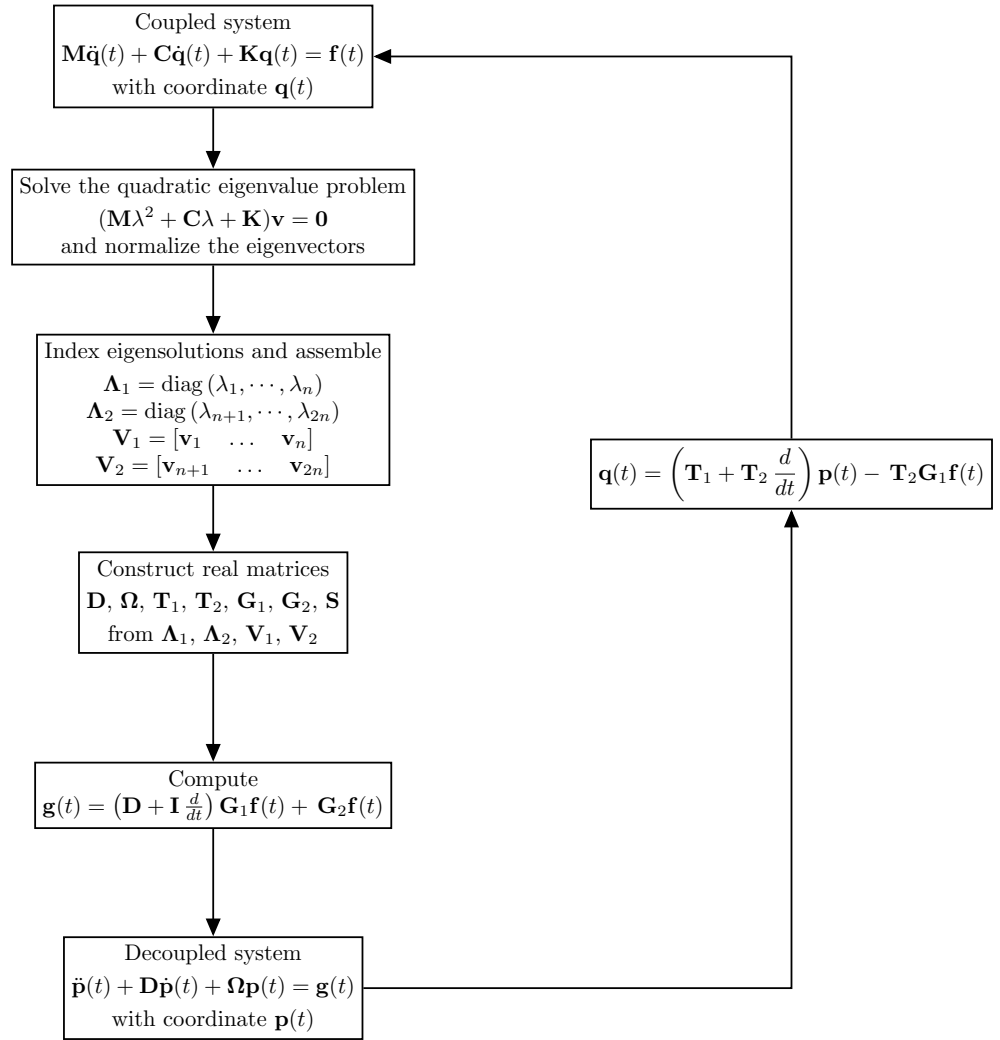
The upper half of equation (2.26) yields a configuration space mapping from  $\mathbf{q}$  to  $\mathbf{p}$  that is an inverse of equation (2.25). When  $t = 0$ , (2.26) generates the initial values  $\mathbf{p}(0)$  and  $\dot{\mathbf{p}}(0)$  of equation (1.3). The decoupling transformations in both the configuration and state spaces are nonlinear for non-homogeneous systems and linear for homogeneous systems. These four tasks are illustrated schematically in figure 3.

### 2.2.1.1 Relation to Classical Modal Analysis

The decoupling procedure expounded earlier is a direct extension of modal analysis. If equation (1.1) represents an undamped passive system with a mass-normalized modal matrix  $\mathbf{U}$ , then the eigenvectors of equation (2.8) are such that  $\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{U}$  up to arbitrary signs in the columns of  $\mathbf{U}$ . Consequently,

$$\mathbf{T}_1 = \mathbf{T}_4 = \mathbf{U}, \quad \mathbf{T}_2 = \mathbf{T}_3 = \mathbf{0}, \quad \mathbf{S}_1 = \mathbf{S}_4 = \mathbf{U}^{-1}, \quad \mathbf{S}_2 = \mathbf{S}_3 = \mathbf{0}. \quad (2.32)$$

In this case, the configuration space decoupling transformation represented by equation (2.25) reduces to the modal transformation  $\mathbf{q} = \mathbf{U}\mathbf{p}$ . With different indexing schemes, phase synchronization generates all possible decoupled forms into which a linear system can be transformed in real space [24, 25].



**Fig. 3** Flowchart for decoupling a second-order linear system by phase synchronization.

### 2.2.1.2 Illustrative Example: a Diagonalizable System

A two-degree-of-freedom system of the form (1.1) is specified by

$$\mathbf{M} = \mathbf{I}, \quad \mathbf{C} = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} -7 & 4 \\ -8 & 1 \end{bmatrix} \quad (2.33)$$

and initial conditions

$$\mathbf{q}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \dot{\mathbf{q}}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \quad (2.34)$$

Since the coefficients  $\mathbf{C}$  and  $\mathbf{K}$  are non-symmetric, there are gyroscopic and circulatory forces in the system. Observe that  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$  are diagonalizable and pairwise commutative, i.e.,  $\mathbf{MC} = \mathbf{CM}$ ,  $\mathbf{KC} = \mathbf{CK}$  and  $\mathbf{MK} = \mathbf{KM}$ . By matrix theory [31],  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$  are simultaneously diagonalizable by a common similarity transformation. The diagonalizing matrix is

$$\mathbf{V} = \begin{bmatrix} 1 & 1 \\ 1+i & 1-i \end{bmatrix}. \quad (2.35)$$

To decouple by similarity transformation, define a coordinate transformation by  $\mathbf{q} = \mathbf{V}\hat{\mathbf{p}}$ . The decoupled system in coordinate  $\hat{\mathbf{p}}$  is

$$\ddot{\hat{\mathbf{p}}} + \mathbf{V}^{-1}\mathbf{C}\mathbf{V}\dot{\hat{\mathbf{p}}} + \mathbf{V}^{-1}\mathbf{K}\mathbf{V}\hat{\mathbf{p}} = \mathbf{0}, \quad (2.36)$$

which is equivalent to

$$\begin{bmatrix} \ddot{\hat{p}}_1 \\ \ddot{\hat{p}}_2 \end{bmatrix} + \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix} \begin{bmatrix} \dot{\hat{p}}_1 \\ \dot{\hat{p}}_2 \end{bmatrix} + \begin{bmatrix} -3+4i & 0 \\ 0 & -3-4i \end{bmatrix} \begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (2.37)$$

The decoupled system, the decoupling transformation, and initial conditions of the decoupled system are all complex. Thus physical insight is greatly diminished by such a decoupling process. Using phase synchronization, the system can be decoupled into real SDOF systems (1.3) by a real linear transformation.

Solution of the quadratic eigenvalue problem (2.8) yields

$$\mathbf{\Lambda}_1 = \begin{bmatrix} -3.0679 + 0.2254i & 0 \\ 0 & 1.0679 + 1.2254i \end{bmatrix}, \quad \mathbf{\Lambda}_2 = \overline{\mathbf{\Lambda}}_1 \quad (2.38)$$

and

$$\mathbf{V}_1 = \begin{bmatrix} -0.0836 + 0.1975i & 0.1950 + 0.4605i \\ -0.2811 + 0.1139i & 0.6555 + 0.2655i \end{bmatrix}, \quad \mathbf{V}_2 = \overline{\mathbf{V}}_1. \quad (2.39)$$

The eigenvectors are normalized in accordance with equations (2.13) and (2.14). Thus, the decoupled system (1.3) is given by

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} 6.1358 & 0 \\ 0 & -2.1358 \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} + \begin{bmatrix} 9.4627 & 0 \\ 0 & 2.6419 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (2.40)$$



The initial conditions of the decoupled system are

$$\mathbf{p}(0) = \begin{bmatrix} -0.3679 \\ 2.4633 \end{bmatrix}, \quad \dot{\mathbf{p}}(0) = \begin{bmatrix} 1.3260 \\ 3.4719 \end{bmatrix}. \quad (2.41)$$

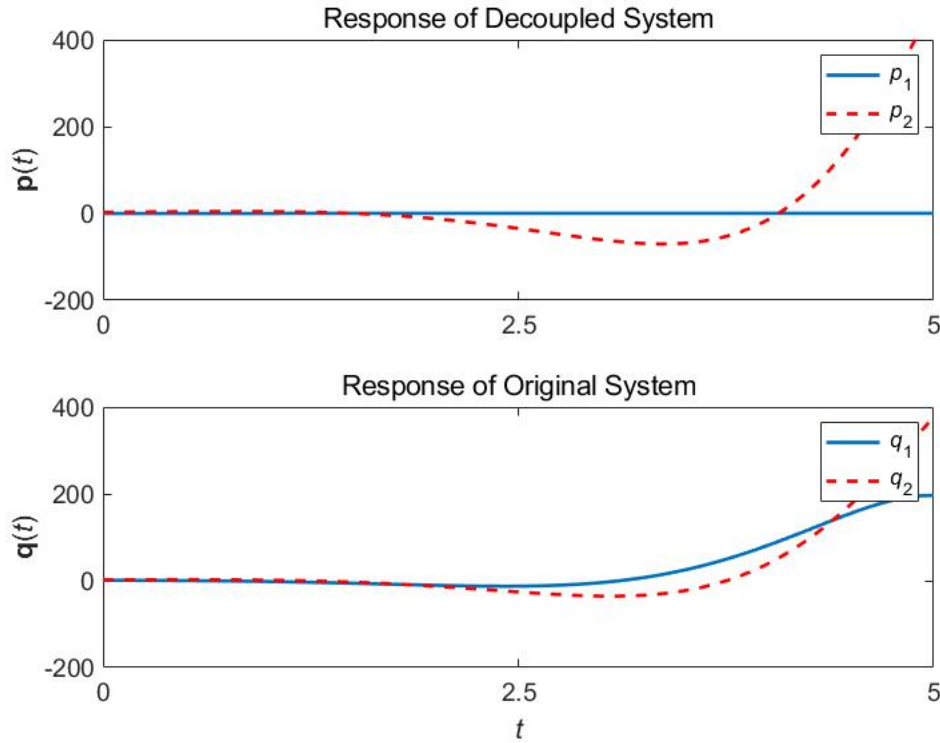
Analytical formulas for the decoupling transformations can be readily constructed. The configuration-space decoupling transformation (2.25) can be expressed as

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 2.6047 & -0.2063 \\ 1.2690 & 0.4241 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 0.8763 & 0.3758 \\ 0.5053 & 0.2167 \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}. \quad (2.42)$$

The state-space form of the decoupling transformation (2.26) is given by

$$\begin{bmatrix} p_1 \\ p_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} -0.3212 & -1.0516 & -0.9946 & 1.0618 \\ -1.1995 & 1.3905 & -0.4266 & 0.4554 \\ 1.5317 & 2.9168 & 2.7970 & -3.2422 \\ 0.6569 & 1.2509 & 0.1377 & 0.4510 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}. \quad (2.43)$$

The response  $\mathbf{p}(t)$  of the decoupled system is compared with the response  $\mathbf{q}(t)$  of the coupled system in figure 4.



**Fig. 4** The response  $\mathbf{q}(t)$  of the non-symmetric coupled system is compared to the response  $\mathbf{p}(t)$  of the decoupled system. Phase synchronization decouples (1.1) into real SDOF systems (1.3).

### 2.2.2 Methodology for Decoupling Defective Systems

In this section, formulas presented previously will be generalized. In addition, any type of linear system not previously considered can be treated by this generalization. When an eigenvalue  $\lambda_j$  of equation (2.8) occurs  $m_j$  times and a full complement of  $m_j$  independent eigenvectors cannot be found, equation (1.1) is defective; the  $\rho_j < m_j$  eigenvectors  $\mathbf{v}_s^j$  ( $s = 1, 2, \dots, \rho_j$ ) must be supplemented by  $m_j - \rho_j$  generalized eigenvectors  $\mathbf{v}_{\rho_j+l}^j$  ( $l = 1, 2, \dots, m_j - \rho_j$ ). These generalized eigenvectors are defined by the sequence [61]

$$\left. \begin{aligned} \mathbf{Q}(\lambda_j)\mathbf{v}_{\rho_j+1}^j + \mathbf{Q}'(\lambda_j)\mathbf{v}_{\rho_j}^j &= \mathbf{0}, \\ \mathbf{Q}(\lambda_j)\mathbf{v}_{\rho_j+2}^j + \mathbf{Q}'(\lambda_j)\mathbf{v}_{\rho_j+1}^j + \frac{1}{2}\mathbf{Q}''(\lambda_j)\mathbf{v}_{\rho_j}^j &= \mathbf{0}, \\ &\vdots \\ \mathbf{Q}(\lambda_j)\mathbf{v}_{m_j}^j + \mathbf{Q}'(\lambda_j)\mathbf{v}_{m_j-1}^j + \frac{1}{2}\mathbf{Q}''(\lambda_j)\mathbf{v}_{m_j-2}^j &= \mathbf{0}, \end{aligned} \right\} \quad (2.44)$$

where

$$\mathbf{Q}(\lambda_j) = \lambda_j^2 \mathbf{M} + \lambda_j^2 \mathbf{C} + \mathbf{K}, \quad \mathbf{Q}'(\lambda_j) = 2\lambda_j \mathbf{M} + \mathbf{C}, \quad \mathbf{Q}''(\lambda_j) = 2\mathbf{M}. \quad (2.45)$$

Once a complete set of vectors is obtained for every defective eigenvalue, it is then possible to convert equation (1.1) into the decoupled system represented by equation (1.3). While defective systems do not typically arise in practical applications, they have received attention from a number of authors [26, 27, 65, 66]. As demonstrated in [26], the decoupling of defective systems is a delicate procedure that can easily vary on a case-by-case basis, but regardless it is always possible to recast equation (1.1) in the decoupled form (1.3).

In general, for homogeneous systems with  $\mathbf{f}(t) = \mathbf{0}$ , equations (1.1) and (1.3) are connected in the state space by a real, invertible and time-varying transformation given by [26]

$$\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_q \\ \mathbf{V}_q \mathbf{J}_q \end{bmatrix} e^{\mathbf{J}_q t} e^{-\mathbf{J}_p t} \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_p \mathbf{J}_p \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \mathbf{T}(t) \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1(t) & \mathbf{T}_2(t) \\ \mathbf{T}_3(t) & \mathbf{T}_4(t) \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix}. \quad (2.46)$$

In the above expression,  $\mathbf{J}_q$  and  $\mathbf{J}_p$  are  $2n \times 2n$  Jordan matrices of the indexed eigenvalues on the diagonal, where  $\mathbf{J}_p$  is usually a modified form of  $\mathbf{J}_q$  whose structure imposes the eigenvalue pairing scheme required for decoupling. The  $n \times 2n$  matrix  $\mathbf{V}_q$  contains the eigenvectors and generalized eigenvectors associated with the indexed eigenvalues in  $\mathbf{J}_q$ , while the structure of the  $n \times 2n$  matrix  $\mathbf{V}_p$  enforces the pairing scheme imposed by  $\mathbf{J}_p$ . The coefficient matrices of equations (1.1) and (1.3) are related by the  $2n \times 2n$  real and invertible transformation matrix  $\mathbf{T}(t)$  according to

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Omega} & -\mathbf{D} \end{bmatrix} = \mathbf{T}^{-1}(t) \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \mathbf{T}(t) - \mathbf{T}^{-1}(t)\dot{\mathbf{T}}(t). \quad (2.47)$$

To decouple equation (1.1) when the excitation is included, consider the state space transformation

$$\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \mathbf{T}(t) \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix}. \quad (2.48)$$

After casting equation (1.1) in the state space as

$$\begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f}(t) \end{bmatrix}, \quad (2.49)$$

substitute equation (2.48) into equation (2.49), pre-multiply the result by  $\mathbf{T}^{-1}(t)$ , and use relationship (2.47) to obtain

$$\begin{bmatrix} \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Omega} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix}, \quad (2.50)$$

where

$$\begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} = \mathbf{T}^{-1}(t) \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{G}_1(t)\mathbf{f}(t) \\ \mathbf{G}_2(t)\mathbf{f}(t) \end{bmatrix}. \quad (2.51)$$

Extracting the upper and lower halves of equation (2.50), eliminating the coordinate  $\mathbf{p}_2$ , and comparing the result to equation (1.3) reveals that

$$\mathbf{p}_1 = \mathbf{p}, \quad \mathbf{p}_2 = \dot{\mathbf{p}} - \mathbf{G}_1(t)\mathbf{f}(t) \quad (2.52)$$

and the excitation

$$\mathbf{g}(t) = \left( \mathbf{D} + \mathbf{I} \frac{d}{dt} \right) \mathbf{G}_1(t)\mathbf{f}(t) + \mathbf{G}_2(t)\mathbf{f}(t). \quad (2.53)$$

Consequently, from equation (2.48), the decoupling transformation in the state space is

$$\begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \mathbf{T}^{-1}(t) \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_1(t)\mathbf{f}(t) \end{bmatrix} \quad (2.54)$$

The corresponding configuration space decoupling transformation is given by

$$\mathbf{q}(t) = \left( \mathbf{T}_1(t) + \mathbf{T}_2(t) \frac{d}{dt} \right) \mathbf{p}(t) - \mathbf{T}_2(t)\mathbf{G}_1(t)\mathbf{f}(t). \quad (2.55)$$

When  $t = 0$ , equation (2.54) generates the initial values  $\mathbf{p}(0)$  and  $\dot{\mathbf{p}}(0)$  of equation (1.3).

Decoupling a defective system represented by equation (1.1) is less systematic than in the non-defective case, as the process for constructing the coefficient matrices  $\mathbf{D}$  and  $\mathbf{\Omega}$  of equation (1.3) varies with the number of real eigenvalues and with the geometric multiplicities of the defective eigenvalues.

### 2.2.2.1 A Possible Simplification

It is generally not possible to simplify the time-varying transformation matrix  $\mathbf{T}(t)$  in equation (2.46) to a more explicit and descriptive form, as exemplified by equation (2.27) when the system is non-defective. However, a special case in which simplification occurs is when all eigenvalues are complex. Suppose  $2N < 2n$  of these eigenvalues are distinct and, for simplicity, each defective eigenvalue has unit geometric multiplicity (i.e., each has one associated eigenvector). The latter assumption is simply a matter of convenience and can be relaxed with care [26]. Let  $m_j$  ( $j = 1, 2, \dots, N$ ) denote the algebraic multiplicity (the number of occurrences) of each unique eigenvalue  $\lambda_j$  with positive imaginary part. Associated with  $\lambda_j$  is an  $m_j \times m_j$  Jordan block

$$\mathbf{J}_j = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_j & 1 \\ 0 & \cdots & 0 & 0 & \lambda_j \end{bmatrix} = \lambda_j \mathbf{I}_{m_j} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} = \mathbf{\Lambda}_j + \mathbf{N}_j. \quad (2.56)$$

Under the assumption of unit geometric multiplicity,  $\lambda_j$  has a single eigenvector  $\mathbf{v}_1^j$  and  $m_j - 1$  generalized eigenvectors  $\mathbf{v}_s^j$  ( $s = 2, 3, \dots, m_j$ ) that are computed according to (2.44) and (2.45). Arrange the vectors in an  $n \times m_j$  matrix

$$\mathbf{V}_j = \begin{bmatrix} \mathbf{v}_1^j & \mathbf{v}_2^j & \cdots & \mathbf{v}_{m_j}^j \end{bmatrix}. \quad (2.57)$$

Compile the  $N$  Jordan blocks  $\mathbf{J}_j$  and the  $N$  matrices  $\mathbf{V}_j$  of eigenvectors and generalized eigenvectors to form the  $n \times n$  matrices

$$\mathbf{J} = \text{diag}(\mathbf{J}_1, \mathbf{J}_2, \dots, \mathbf{J}_N), \quad \mathbf{V} = [\mathbf{V}_1 \quad \mathbf{V}_2 \quad \cdots \quad \mathbf{V}_N]. \quad (2.58)$$

Likewise, construct the following  $n \times n$  matrices from the  $N$  diagonal matrices  $\mathbf{\Lambda}_j$  and the  $N$  nilpotent matrices  $\mathbf{N}_j$  defined in equation (2.56)

$$\mathbf{\Lambda} = \text{diag}(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \dots, \mathbf{\Lambda}_N), \quad \mathbf{N} = \text{diag}(\mathbf{N}_1, \mathbf{N}_2, \dots, \mathbf{N}_N). \quad (2.59)$$

Note that the matrices  $\mathbf{\Lambda}$  and  $\mathbf{N}$  commute in multiplication. For this special case of a defective system, the decoupling transformation is such that [26]

$$\mathbf{J}_q = \text{diag}(\mathbf{J}, \bar{\mathbf{J}}), \quad \mathbf{V}_q = [\mathbf{V} \quad \bar{\mathbf{V}}], \quad \mathbf{J}_p = \text{diag}(\mathbf{\Lambda}, \bar{\mathbf{\Lambda}}), \quad \mathbf{V}_p = [\mathbf{I} \quad \bar{\mathbf{I}}], \quad (2.60)$$

where the coefficient matrices of the decoupled equation (1.3) are given by

$$\mathbf{D} = -(\mathbf{\Lambda} + \bar{\mathbf{\Lambda}}), \quad \mathbf{\Omega} = \mathbf{\Lambda} \bar{\mathbf{\Lambda}}. \quad (2.61)$$

In other words, equation (1.3) comprises  $N$  collections of  $m_j$  identical, independent SDOF systems with generally different excitations and initial values. Based on equation (2.60), the state transformation matrix  $\mathbf{T}(t)$  defined in equation (2.46) becomes

$$\mathbf{T}(t) = \begin{bmatrix} \mathbf{T}_1(t) & \mathbf{T}_2(t) \\ \mathbf{T}_3(t) & \mathbf{T}_4(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \bar{\mathbf{V}} \\ \mathbf{VJ} & \bar{\mathbf{VJ}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \bar{\mathbf{\Lambda}} & \bar{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} e^{\mathbf{N}t} & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{N}t} \end{bmatrix}, \quad (2.62)$$

where the  $n \times n$  sub-matrices

$$\mathbf{T}_1(t) = (\mathbf{V}\bar{\mathbf{\Lambda}} - \bar{\mathbf{V}}\mathbf{\Lambda}) (\bar{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} e^{\mathbf{N}t}, \quad (2.63)$$

$$\mathbf{T}_2(t) = (\bar{\mathbf{V}} - \mathbf{V}) (\bar{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} e^{\mathbf{N}t}, \quad (2.64)$$

$$\mathbf{T}_3(t) = [(\mathbf{VJ})\bar{\mathbf{\Lambda}} - (\bar{\mathbf{VJ}})\mathbf{\Lambda}] (\bar{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} e^{\mathbf{N}t}, \quad (2.65)$$

$$\mathbf{T}_4(t) = (\bar{\mathbf{VJ}} - \mathbf{VJ}) (\bar{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} e^{\mathbf{N}t}. \quad (2.66)$$

As a result, equation (2.51) yields

$$\mathbf{G}_1(t) = e^{-\mathbf{N}t} \left\{ [\mathbf{VJ} - (\bar{\mathbf{VJ}})\bar{\mathbf{V}}^{-1}\mathbf{V}]^{-1} + [(\bar{\mathbf{VJ}}) - \mathbf{VJ}\mathbf{V}^{-1}\bar{\mathbf{V}}]^{-1} \right\} \mathbf{M}^{-1} \quad (2.67)$$

and

$$\mathbf{G}_2(t) = e^{-\mathbf{N}t} \left\{ \mathbf{\Lambda} [\mathbf{VJ} - (\bar{\mathbf{VJ}})\bar{\mathbf{V}}^{-1}\mathbf{V}]^{-1} + \bar{\mathbf{\Lambda}} [(\bar{\mathbf{VJ}}) - \mathbf{VJ}\mathbf{V}^{-1}\bar{\mathbf{V}}]^{-1} \right\} \mathbf{M}^{-1} \quad (2.68)$$

It is generally not possible to express the transformation matrix  $\mathbf{T}(t)$  in an explicit form such as equation (2.62) when some of the defective eigenvalues are real. Additional details of the decoupling of equation (1.1) when it possesses defective real eigenvalues are provided in [26].

The inverse of (2.62) involves a matrix  $\mathbf{S}$ , which is also time-varying and is defined as follows:

$$\mathbf{S}(t) = \begin{bmatrix} \mathbf{S}_1(t) & \mathbf{S}_2(t) \\ \mathbf{S}_3(t) & \mathbf{S}_4(t) \end{bmatrix} = \mathbf{T}^{-1}(t) = \begin{bmatrix} e^{-\mathbf{N}t} & \mathbf{0} \\ \mathbf{0} & e^{-\mathbf{N}t} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \bar{\mathbf{\Lambda}} & \bar{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \bar{\mathbf{V}} \\ \mathbf{VJ} & \bar{\mathbf{VJ}} \end{bmatrix}^{-1}, \quad (2.69)$$

where the  $n \times n$  sub-matrices

$$\mathbf{S}_1(t) = e^{-\mathbf{N}t} \left[ (\mathbf{VJ})^{-1} - (\bar{\mathbf{VJ}})^{-1} \right] \left[ \mathbf{V}(\mathbf{VJ})^{-1} - \bar{\mathbf{V}}(\bar{\mathbf{VJ}})^{-1} \right]^{-1}, \quad (2.70)$$

$$\mathbf{S}_2(t) = e^{-\mathbf{N}t} \left[ \mathbf{V}^{-1} - \bar{\mathbf{V}}^{-1} \right] \left[ (\mathbf{VJ})\mathbf{V}^{-1} - (\bar{\mathbf{VJ}})\bar{\mathbf{V}}^{-1} \right]^{-1}, \quad (2.71)$$

$$\mathbf{S}_3(t) = e^{-\mathbf{N}t} \left[ \mathbf{\Lambda}(\mathbf{VJ})^{-1} - \bar{\mathbf{\Lambda}}(\bar{\mathbf{VJ}})^{-1} \right] \left[ \mathbf{V}(\mathbf{VJ})^{-1} - \bar{\mathbf{V}}(\bar{\mathbf{VJ}})^{-1} \right]^{-1}, \quad (2.72)$$

$$\mathbf{S}_4(t) = e^{-\mathbf{N}t} \left[ \mathbf{\Lambda}\mathbf{V}^{-1} - \bar{\mathbf{\Lambda}}\bar{\mathbf{V}}^{-1} \right] \left[ (\mathbf{VJ})\mathbf{V}^{-1} - (\bar{\mathbf{VJ}})\bar{\mathbf{V}}^{-1} \right]^{-1}. \quad (2.73)$$

Should this system be non-defective, then the matrices  $\mathbf{N} = \mathbf{0}$  and  $\mathbf{J} = \mathbf{\Lambda}$ . Taking  $\mathbf{\Lambda} = \mathbf{\Lambda}_1$ ,  $\bar{\mathbf{\Lambda}} = \mathbf{\Lambda}_2$ ,  $\mathbf{V} = \mathbf{V}_1$  and  $\bar{\mathbf{V}} = \mathbf{V}_2$ , it is easy to verify that all formulae for transforming equation (1.1) into the form (1.3) reduce to their non-defective counterparts.

### 2.2.2.2 Illustrative Example: a Defective System

A non-classically damped, two-degree-of-freedom system is specified by

$$\mathbf{M} = \mathbf{I}, \quad \mathbf{C} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}. \quad (2.74)$$

The initial conditions are prescribed as

$$\mathbf{q}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \dot{\mathbf{q}}(0) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}. \quad (2.75)$$

Solving the associated quadratic eigenvalue problem (2.8), it is found that there is one pair of defective complex conjugate eigenvalues  $\lambda_1 = -1 + i\sqrt{2}$  of algebraic multiplicity  $m_1 = 2$  and unit geometric multiplicity ( $\rho_1 = 1$ ). The sole eigenvector is

$$\mathbf{v}_1^1 = \begin{bmatrix} -i\sqrt{2} \\ 1 \end{bmatrix}, \quad (2.76)$$

which has not been subjected to any normalization scheme. The additional generalized eigenvector, computed according to (2.44) and (2.45), is

$$\mathbf{v}_2^1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}. \quad (2.77)$$

Pairing of the complex conjugate eigenvalues implies that, from equation (2.56),

$$\mathbf{\Lambda} = (-1 + i\sqrt{2})\mathbf{I}. \quad (2.78)$$

It follows from equations (2.56), (2.57) and (2.58) that the matrices  $\mathbf{J}$ ,  $\mathbf{N}$ , and  $\mathbf{V}$  are of the form

$$\mathbf{J} = \begin{bmatrix} -1 + i\sqrt{2} & 1 \\ 0 & -1 + i\sqrt{2} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} -i\sqrt{2} & 3 \\ 1 & 0 \end{bmatrix}. \quad (2.79)$$

By equations (2.63) and (2.64), the transformation matrices  $\mathbf{T}_1$  and  $\mathbf{T}_2$  are

$$\mathbf{T}_1 = \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{T}_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (2.80)$$

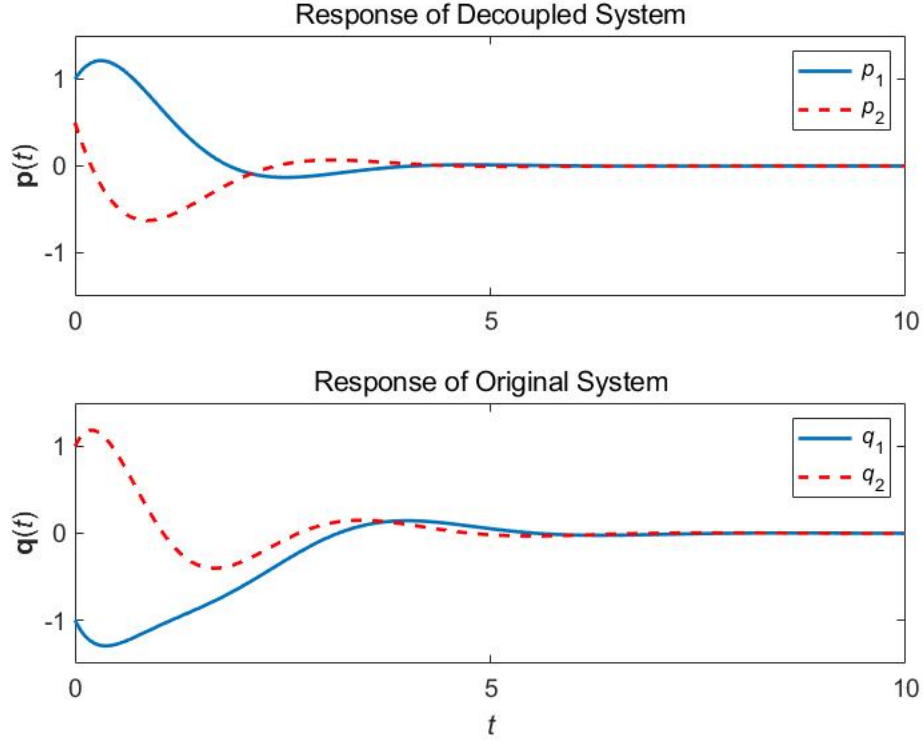
The decoupled system's coefficient matrices are given by (2.61)

$$\mathbf{D} = 2\mathbf{I}, \quad \mathbf{\Omega} = 3\mathbf{I}. \quad (2.81)$$

This implies that the decoupled system consists of two identical underdamped SDOF oscillators. However, the initial conditions for each decoupled coordinate  $p_j(t)$  ( $j = 1, 2$ ) are different:

$$\mathbf{p}(0) = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad \dot{\mathbf{p}}(0) = \begin{bmatrix} 1.5 \\ -3 \end{bmatrix}. \quad (2.82)$$

The response  $\mathbf{p}(t)$  of the decoupled system is compared with the response  $\mathbf{q}(t)$  of the coupled system in figure 5.



**Fig. 5** The response  $\mathbf{q}(t)$  of the coupled defective system and the response  $\mathbf{p}(t)$  of the decoupled system.

### 2.2.3 Phase Synchronization as Isospectral Transformation

As illustrated in the example 2.2.2.2, while the coupled system has a defective complex conjugate pair, the decoupled system was formed by two equal damped oscillators, i.e., the decoupled system was not defective. This happened because a SDOF system cannot have a defective complex eigenvalue.

The limitations on the spectrum of a SDOF system makes it not possible for the entirety of the spectrum of the coupled system (1.1) to be represented by the decoupled system (1.3). Consider the homogeneous part of the  $j$ th-component equation of (1.3):

$$\ddot{p}_j - (\lambda_j + \lambda_{n+j})\dot{p}_j + \lambda_j\lambda_{n+j}p_j = 0. \quad (2.83)$$

This SDOF system has real coefficients and the roots of its characteristic polynomial  $\lambda_j$  and  $\lambda_{n+j}$  are either complex conjugates, distinct and real, or defective and real (forming a Jordan block of size  $2 \times 2$ ) [2–16]. Therefore a coupled system that has a defective complex eigenvalue cannot have a decoupled system with this eigenstructure. The best one can do in this case

is to decouple with the eigenvalues preserved, but not their partial multiplicities (see other examples in [26]). The following theorem, proved in [27], states the conditions under which decoupling can be obtained while preserving the original system's eigenstructure:

**Theorem 2.1.** *Any real second-order linear system (1.1) with nonzero leading coefficient and whose associated spectrum satisfies*

- i all nonreal eigenvalues are semisimple and occur in conjugate pairs;*
  - ii all Jordan blocks associated with a defective eigenvalue are no larger than  $2 \times 2$ ;*
  - iii excluding the nonreal eigenvalues and the  $2 \times 2$  Jordan blocks, all remaining real eigenvalues, which have unit partial multiplicities, form pairs of distinct eigenvalues;*
- can be converted into a real diagonal form (1.3) of the same dimension by a linear time-invariant transformation that preserves the eigenvalues and their partial multiplicities.*

If these conditions are not met, one can still decouple by only preserving the algebraic multiplicities of the eigenvalues. Perhaps this is better illustrated by an example.

### 2.2.3.1 Illustrative Example: Isospectral Decoupling

Consider the two-degree-of-freedom system

$$\ddot{\mathbf{q}} + \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 7 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{q} = \mathbf{0}. \quad (2.84)$$

Solution of the quadratic eigenvalue problem (2.8) reveals that the spectrum consists of two simple real eigenvalues  $\lambda_1 = -3$  and  $\lambda_2 = -2$  with associated eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} -2.5 \\ 1.25 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ -4 \end{bmatrix}, \quad (2.85)$$

respectively. The eigenvalue  $\lambda_3 = -1$ , repeated two times, is defective with unit geometric multiplicity and its eigenvector is

$$\mathbf{v}_3^1 = \begin{bmatrix} 0 \\ -1.5 \end{bmatrix}. \quad (2.86)$$

The associated generalized eigenvector  $\mathbf{v}_3^2$ , given by (2.44) and (2.45), is

$$\mathbf{v}_3^2 = \begin{bmatrix} -0.5 \\ 2.75 \end{bmatrix}. \quad (2.87)$$

No particular normalization was used for the eigenvectors. The spectrum information for the coupled system is stored in the following matrices:

$$\mathbf{J}_q = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}, \quad \mathbf{V}_q = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3^1 \quad \mathbf{v}_3^2]. \quad (2.88)$$



System (2.84) and its decoupled form are related by the linear time-varying state space transformation (2.46) and the initial conditions of the decoupled system are given by (2.54) by setting  $t = 0$ . It is possible to decouple system (2.84) in two ways: isospectrally or by preserving its eigenvalues and not their partial multiplicities; the differences between these two approaches appear mainly in  $\mathbf{J}_p$  and  $\mathbf{V}_p$ .

For isospectral decoupling, the same eigenstructure must be imposed on the decoupled system. This means  $\mathbf{J}_p = \mathbf{J}_q$ . The pairing scheme is such that the defective real eigenvalue must be paired with itself. It leaves the remaining distinct real eigenvalues to form a pair. This is enforced by the matrix

$$\mathbf{V}_p = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad (2.89)$$

Therefore,

$$\mathbf{\Lambda}_1 = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{\Lambda}_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \quad (2.90)$$

and the decoupled system is

$$\ddot{\mathbf{p}} + \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \dot{\mathbf{p}} + \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{p} = \mathbf{0}. \quad (2.91)$$

The decoupling transformation is then given by

$$\begin{aligned} \mathbf{q}(t) &= \left( \mathbf{T}_1 + \mathbf{T}_2 \frac{d}{dt} \right) \mathbf{p}(t) \\ &= \begin{bmatrix} 17 & -0.5 \\ -14.5 & 1.25 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 6.5 & -0.5 \\ -5.25 & 2.75 \end{bmatrix} \begin{bmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \end{bmatrix}. \end{aligned} \quad (2.92)$$

The second possible method for decoupling is attained by enforcing that only the eigenvalues be preserved, not their partial multiplicities. Here, each distinct eigenvalue is paired with the defective eigenvalue as

$$\mathbf{\Lambda}_1 = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{\Lambda}_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}. \quad (2.93)$$

The decoupled system is

$$\ddot{\mathbf{p}} + \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \dot{\mathbf{p}} + \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{p} = \mathbf{0}. \quad (2.94)$$

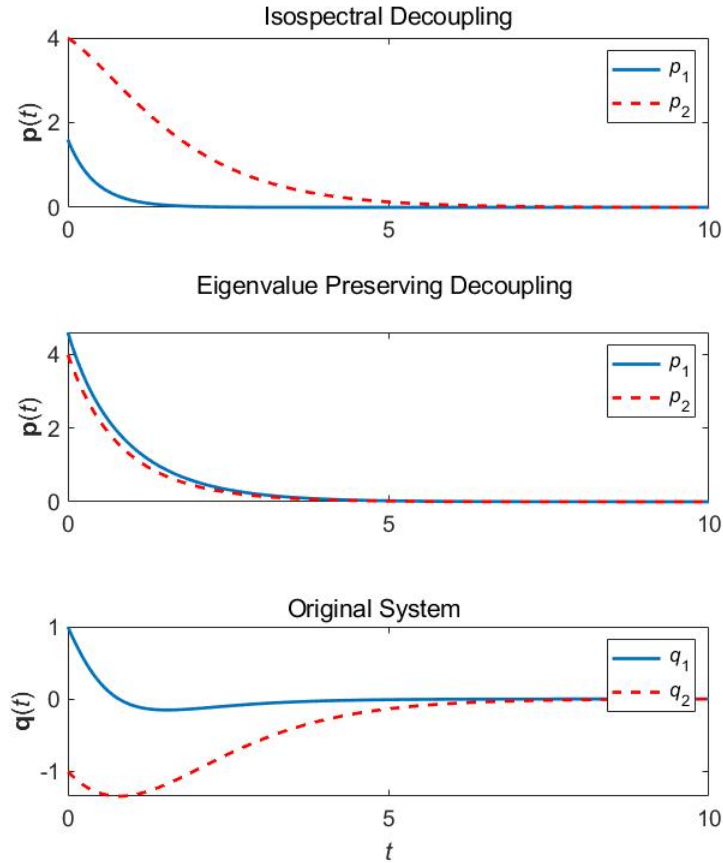
To achieve this, let

$$\mathbf{J}_p = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}, \quad \mathbf{V}_p = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \quad (2.95)$$

In this case the decoupling transformation is time-varying and given by

$$\begin{aligned} \mathbf{q}(t) &= \left( \mathbf{T}_1 + \mathbf{T}_2 \frac{d}{dt} \right) \mathbf{p}(t) \\ &= \begin{bmatrix} 1.25 & -5 \\ -2.875 & 9.5 - 3t \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 1.25 & -4.5 \\ -1.375 & 6.75 - 1.5t \end{bmatrix} \begin{bmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \end{bmatrix}. \end{aligned} \quad (2.96)$$

While isospectral decoupling is only possible when the conditions in theorem 2.1 are met, eigenvalue preserving decoupling is always permissible. Figure 6 shows the response of the decoupled systems obtained by the two methods and original system.



**Fig. 6** The response  $\mathbf{q}(t)$  of the coupled defective system is compared with the response of the decoupled systems obtained by isospectral decoupling and eigenvalue preserving decoupling. The initial conditions used for the simulation are  $\mathbf{q}(0) = [1 \quad -1]^T$  and  $\dot{\mathbf{q}}(0) = [-2 \quad -1]^T$ .

## Chapter 3

# A Canonical Form of the Equation of Motion

It is well known that the equation of motion of a SDOF passive system can be converted into an undamped system by an invertible transformation. For a MDOF passive system, this reduction poses a challenge because the equation of motion is usually coupled. The reduction is still permissible under the assumption of classical damping, whereby a passive system can be decoupled by modal analysis into a series of independent SDOF systems. However, passive systems are non-classically damped in general. In this chapter it will be shown that linear systems can be transformed so as to eliminate the coefficient of velocity from their equations of motion. In addition, the remaining two coefficient matrices can be reduced to diagonal forms. The original impetus was to show that any passive system can be transformed into an undamped one, an important result that has become an offshoot.

The organization of the chapter is as follows. In section 3.1, the reduction of the equation of motion to a canonical form specified by two diagonal matrices is formulated in mathematical terms and previously known results are reviewed. This is followed by an explicit transformation to generate the canonical form of the equation of motion of non-defective systems in section 3.2.1. The reduction of defective linear systems is treated in section 3.2.2. A summary of findings is provided in section 3.3. Two numerical examples are supplied for illustration.

### 3.1 Problem Formulation

It will be shown that equation (1.1) can be reduced, by an invertible transformation, to the real decoupled form

$$\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} = \mathbf{h}(t) \quad (3.1)$$

where  $\mathbf{B}$  is a diagonal matrix, and the generalized displacement  $\mathbf{x}$  and excitation  $\mathbf{h}(t)$  are  $n$ -dimensional column vectors. Basically, a transformation will be found to convert  $\mathbf{M}$  and

$\mathbf{K}$  into diagonal matrices while removing  $\mathbf{C}\dot{\mathbf{q}}$  at the same time. The canonical form specified by equation (3.1) is the simplest representation of linear dynamical systems.

### 3.1.1 SDOF Systems

It is well known that passive systems either of a single degree or under classical damping can be reduced to an undamped form. Should equation (1.1) represent a SDOF system, it may be rewritten as

$$m\ddot{q} + c\dot{q} + kq = f(t), \quad (3.2)$$

where  $m$ ,  $c$  and  $k$  are just real numbers. Using the invertible transformation [5, pp. 558-559, 36, pp. 394-395, 37, pp. 95-96, 38, pp. 42-43]

$$x = \exp\left(\frac{c}{2m}t\right) q, \quad (3.3)$$

it can be readily verified that equation (3.2) is converted into

$$\ddot{x} + \left(\frac{k}{m} - \frac{c^2}{4m^2}\right) x = \frac{1}{m} \exp\left(\frac{c}{2m}t\right) f(t). \quad (3.4)$$

This undamped form is sometimes referred to as the normal form of a SDOF system. Perhaps it would not be surprising that transformation to an undamped form involves an exponential factor. In free vibration, the response  $q(t)$  decays exponentially with any amount of viscous damping. This decay is arrested by the exponential term in equation (3.3), which also exponentially magnifies the excitation of  $x(t)$  in equation (3.4).

### 3.1.2 Classically Damped Systems

If a passive system is classically damped, then it can be decoupled by modal analysis into a series of independent SDOF systems. A classically damped passive system represented by equation (1.1) can be converted into (1.3) for which the modal damping matrix  $\mathbf{D} = \mathbf{U}^T \mathbf{C} \mathbf{U}$  is diagonal. To eliminate the damping term in equation (1.3), apply the transformation

$$\mathbf{x} = \exp\left(\frac{1}{2}\mathbf{D}t\right) \mathbf{p}. \quad (3.5)$$

Observe that  $\exp(\mathbf{D}t/2)$  is a diagonal matrix. Upon transformation, equation (1.3) is converted into

$$\ddot{\mathbf{x}} + \left(\mathbf{\Omega} - \frac{1}{4}\mathbf{D}^2\right) \mathbf{x} = \exp\left(\frac{\mathbf{D}t}{2}\right) \mathbf{U}^T \mathbf{f}(t). \quad (3.6)$$

The original system becomes undamped and decoupled with respect to the coordinate  $\mathbf{x}$ , which is connected with  $\mathbf{q}$  by

$$\mathbf{q} = \mathbf{U} \exp\left(-\frac{1}{2}\mathbf{D}t\right) \mathbf{x}. \quad (3.7)$$

The key to successful reduction of the equation of motion of classically damped linear systems, as described earlier, is decoupling in real space. In general, there is no reason why equation (2.7) should be satisfied for modal analysis to be applicable. In this connection, an attempt was made to reduce the equation of motion of damped gyroscopic systems, for which only the coefficient matrix  $\mathbf{C}$  is non-symmetric, to a form devoid of the velocity term [28]. In addition, the possibility of decoupling equation (1.1) by a time-invariant linear transformation (analogous to modal analysis) was examined [19]. It has been found that a condition equivalent to equation (2.7) is required in both cases.

## 3.2 Generation of the canonical form

An explicit transformation is developed in this section to convert equation (1.1) into the canonical form specified by equation (3.1).

### 3.2.1 Distinct Eigenvalues

When the eigenvalues of equation (2.8) are distinct, equation (1.1) can be decoupled into equation (1.3) by either the configuration space transformation (2.25) or the state space transformation (2.26). To eliminate the velocity term in equation (1.3), apply the transformation

$$\mathbf{x} = \exp\left(\frac{1}{2}\mathbf{D}t\right)\mathbf{p}. \quad (3.8)$$

Upon transformation, equation (1.3) is converted into equation (3.1), for which

$$\mathbf{B} = \mathbf{\Omega} - \frac{1}{4}\mathbf{D}^2 \quad (3.9)$$

is a real diagonal matrix, and

$$\mathbf{h}(t) = \exp\left(\frac{1}{2}\mathbf{D}t\right)\mathbf{g}(t) = \exp\left(\frac{1}{2}\mathbf{D}t\right)\left\{\left(\mathbf{D} + \mathbf{I}\frac{d}{dt}\right)\mathbf{G}_1\mathbf{f}(t) + \mathbf{G}_2\mathbf{f}(t)\right\}. \quad (3.10)$$

Consequently, when recast in the generalized coordinate  $\mathbf{x}$ , equation (1.1) takes on a decoupled form devoid of the velocity term. To determine a configuration space transformation between  $\mathbf{q}$  and  $\mathbf{x}$ , combine equations (2.25) and (3.8) to yield

$$\mathbf{q} = \left(\mathbf{T}_1 + \mathbf{T}_2\frac{d}{dt}\right)\exp\left(-\frac{1}{2}\mathbf{D}t\right)\mathbf{x} - \mathbf{T}_2\mathbf{G}_1\mathbf{f}(t). \quad (3.11)$$

Alternatively, a state space transformation can be determined by combining equations (2.26) and (3.8) to obtain

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \exp\left(\frac{1}{2}\mathbf{D}t\right) & \mathbf{0} \\ \frac{1}{2}\mathbf{D}\exp\left(\frac{1}{2}\mathbf{D}t\right) & \exp\left(\frac{1}{2}\mathbf{D}t\right) \end{bmatrix} \left\{ \mathbf{S} \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_1\mathbf{f}(t) \end{bmatrix} \right\}. \quad (3.12)$$

When  $t = 0$ , the above state space transformation generates the initial values  $\mathbf{x}(0)$  and  $\dot{\mathbf{x}}(0)$  of the canonical form (3.1). The transformations given by equations (3.11) and (3.12) are both real, nonlinear and invertible. In the reduction of equation (1.1), the canonical form specified by equation (3.1) is the simplest representation that one may achieve.

The generation of the canonical form defined by equation (3.1) is certainly applicable to passive systems, which are characterized by three symmetric and positive definite coefficient matrices. Consequently, a solution to the well-trodden problem of reducing a damped linear system to an undamped form has been provided herein.

### 3.2.1.1 Uniqueness of the Canonical Form

How many different canonical forms, of the type defined by equation (3.1) into which equation (1.1) can be reduced, are there? It is obvious that the canonical form (3.1) is unique if phase synchronization generates a unique decoupled system represented by equation (1.3). However, phase synchronization can be implemented with different indexing and normalization schemes. For a given indexing scheme, the coefficient matrices  $\mathbf{D}$  and  $\mathbf{\Omega}$  of equation (1.3) are independent of the normalization of eigenvectors because they are constructed from the eigenvalues. As a result, the homogeneous part of equation (3.1) remains unchanged by normalization. By contrast, the excitation  $\mathbf{h}(t)$  of equation (3.1) is dependent on the eigenvectors of equation (2.8) and, therefore, on the normalization used. However, normalization has no physical significance and is just a matter of convenience. For a given indexing scheme, the canonical form (3.1) is unique up to the normalization of eigenvectors.

There remains the question of equivalence due to different indexing schemes. Two decoupled systems are regarded as the same if their component equations coincide; the order in which the component equations appear is immaterial. Hence, indexing schemes that re-order the component equations of equation (3.1) are considered equivalent. Any indexing scheme must pair the complex conjugate eigensolutions. For a given normalization scheme, there is only one decoupled system associated with equation (1.3) if all eigenvalues are complex, and, therefore, only one canonical form defined by equation (3.1). If there are  $2n_r$  distinct real eigenvalues of equation (2.8), then there are

$$N_r = \frac{\binom{2n_r}{2} \binom{2n_r-2}{2} \binom{2n_r-4}{2} \cdots \binom{2}{2}}{n_r!} = \frac{(2n_r)!}{2^{n_r} n_r!} \quad (3.13)$$

different ways to pair the real eigensolutions [24]. Indeed, using a fixed normalization but different indexing schemes, there are  $N_r$  different decoupled systems associated with equation (1.3), and hence  $N_r$  different canonical forms defined by equation (3.1). These  $N_r$  canonical forms usually have different homogeneous parts. For a non-defective system with repeated eigenvalues, the number of different canonical forms is less than  $N_r$ . It can be stated that various indexing and normalization schemes generate an equivalence class of canonical forms of the type defined by equation (3.1). However, there are not more than  $N_r$  members of this

equivalence class that are essentially different with different homogeneous parts.

### 3.2.1.2 Illustrative Example: a Non-symmetric System

Consider a two-degree-of-freedom system governed by

$$\ddot{\mathbf{q}} + \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.3 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.4 \end{bmatrix} \mathbf{q} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin 2t, \quad (3.14)$$

with initial values  $\mathbf{q}(0) = \mathbf{0}$  and  $\dot{\mathbf{q}}(0) = \mathbf{0}$ . This is a realization of equation (1.1) with non-symmetric coefficient matrices. Solution of the quadratic eigenvalue problem (2.8) yields

$$\mathbf{\Lambda}_1 = \begin{bmatrix} -0.0402 + 0.3683i & 0 \\ 0 & -0.1598 + 0.9599i \end{bmatrix}, \quad \mathbf{\Lambda}_2 = \overline{\mathbf{\Lambda}}_1 \quad (3.15)$$

and

$$\mathbf{V}_1 = \begin{bmatrix} 0.4756 + 0.1059i & 0.7404 - 0.0497i \\ -0.9092 + 0.0139i & 0.6698 + 0.0632i \end{bmatrix}, \quad \mathbf{V}_2 = \overline{\mathbf{V}}_1. \quad (3.16)$$

The eigenvectors are normalized in accordance with equations (2.13) and (2.14). Since all eigenvalues are complex and distinct, there is only one canonical form of the type defined by equation (3.1), unique up to the normalization of eigenvectors. The real and diagonal coefficients of the decoupled equation (1.3) are given by

$$\mathbf{D} = \begin{bmatrix} 0.0804 & 0 \\ 0 & 0.3196 \end{bmatrix}, \quad \mathbf{\Omega} = \begin{bmatrix} 0.1373 & 0 \\ 0 & 0.9470 \end{bmatrix}. \quad (3.17)$$

From equations (2.19) and (2.20),

$$\mathbf{G}_1 = \begin{bmatrix} -0.0374 & 0.1698 \\ -0.1770 & 0.2236 \end{bmatrix}, \quad \mathbf{G}_2 = \begin{bmatrix} 0.6808 & -0.7721 \\ 0.9234 & 0.4728 \end{bmatrix}. \quad (3.18)$$

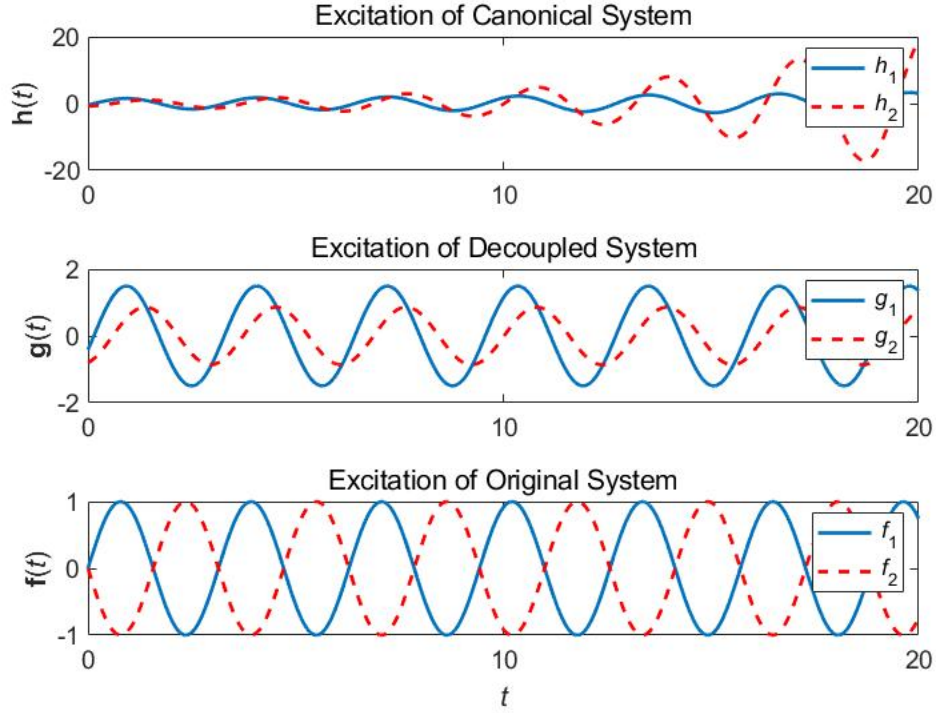
It can be verified that the canonical form (3.1) is specified by

$$\ddot{\mathbf{x}} + \begin{bmatrix} 0.1357 & 0 \\ 0 & 0.9215 \end{bmatrix} \mathbf{x} = \mathbf{h}(t), \quad (3.19)$$

for which

$$\mathbf{h}(t) = \begin{bmatrix} (-0.4144 \cos 2t + 1.4362 \sin 2t)e^{0.0402t} \\ (-0.8012 \cos 2t + 0.3226 \sin 2t)e^{0.1598t} \end{bmatrix}. \quad (3.20)$$

Figure 7 shows the excitation of the original, decoupled and canonical systems.



**Fig. 7** The excitation  $\mathbf{h}(t)$  of the canonical system (3.19), the excitation  $\mathbf{g}(t)$  of the decoupled system with coefficients (3.17) and the excitation  $\mathbf{f}(t)$  of the original system (3.14) are shown.

Using equation (3.11), the configuration space transformation between  $\mathbf{q}$  and  $\mathbf{x}$  can be expressed as

$$\mathbf{q} = \left( \mathbf{E}(t) + \mathbf{F}(t) \frac{d}{dt} \right) \mathbf{x} + \begin{bmatrix} 0.0388 \\ 0.0342 \end{bmatrix} \sin 2t, \quad (3.21)$$

where

$$\mathbf{E}(t) = \begin{bmatrix} 0.4756e^{-0.0402t} & 0.7404e^{-0.1598t} \\ -0.9092e^{-0.0402t} & 0.6698e^{-0.1598t} \end{bmatrix} \quad (3.22)$$

and

$$\mathbf{F}(t) = \begin{bmatrix} 0.2874e^{-0.0402t} & -0.0518e^{-0.1598t} \\ 0.0377e^{-0.0402t} & 0.0658e^{-0.1598t} \end{bmatrix}. \quad (3.23)$$

The state space transformation that reduces equation (3.14) to equation (3.19) is given by equation (3.12), for which

$$\begin{bmatrix} \exp\left(\frac{1}{2}\mathbf{D}t\right) & \mathbf{0} \\ \frac{1}{2}\mathbf{D}\exp\left(\frac{1}{2}\mathbf{D}t\right) & \exp\left(\frac{1}{2}\mathbf{D}t\right) \end{bmatrix} = \begin{bmatrix} e^{0.0402t} & 0 & 0 & 0 \\ 0 & e^{0.1598t} & 0 & 0 \\ 0.0402e^{0.0402t} & 0 & e^{0.0402t} & 0 \\ 0 & 0.1598e^{0.1598t} & 0 & e^{0.1598t} \end{bmatrix}, \quad (3.24)$$

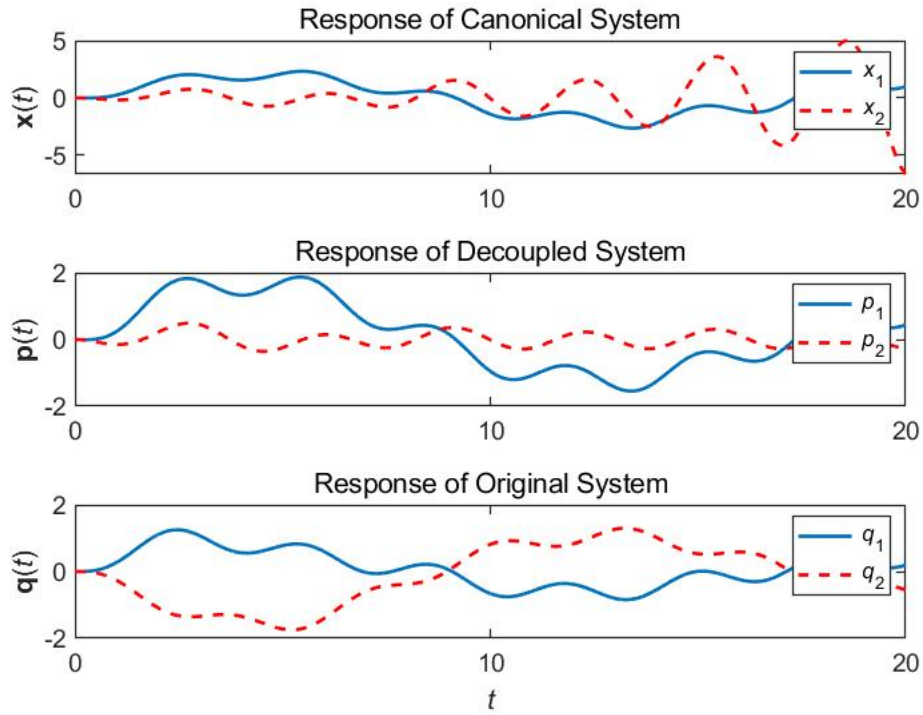


$$\mathbf{S} = \begin{bmatrix} 0.6941 & -0.7286 & -0.0374 & 0.1698 \\ 0.9281 & 0.5045 & -0.1770 & 0.2236 \\ -0.0587 & -0.0567 & 0.6808 & -0.7721 \\ 0.0121 & -0.0363 & 0.9234 & 0.4728 \end{bmatrix} \quad (3.25)$$

and

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{G}_1 \mathbf{f}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.2072 \sin 2t \\ -0.4006 \sin 2t \end{bmatrix}. \quad (3.26)$$

The initial values of equation (3.19) are  $\mathbf{x}(0) = \mathbf{0}$  and  $\dot{\mathbf{x}}(0) = \mathbf{0}$ . Figure 8 shows the response of the original, decoupled and canonical systems. The response  $\mathbf{x}(t)$  of the canonical system (3.19) grows exponentially due to the excitation  $\mathbf{h}(t)$  in (3.20).



**Fig. 8** The responses of the canonical system (3.19), the decoupled system with coefficients (3.17) and the original system (3.14) are shown.

To examine the effect of normalization, let the eigenvectors  $\mathbf{v}_j$  ( $j = 1, 2$ ) be normalized in such a way that the state eigenvectors  $[\mathbf{v}_j \quad \lambda_j \mathbf{v}_j]^T$  have unit Euclidean norm. In this case,

$$\mathbf{V}_1 = \begin{bmatrix} 0.4308 + 0.1028i & 0.5309 \\ -0.8265 & 0.4751 + 0.0773i \end{bmatrix}, \quad \mathbf{V}_2 = \overline{\mathbf{V}_1}. \quad (3.27)$$

The homogeneous part of equation (3.19) remains unchanged because it is constructed from the eigenvalues. However, the excitation  $\mathbf{h}(t)$  in equation (3.19) becomes

$$\mathbf{h}(t) = \begin{bmatrix} (-0.5877 \cos 2t + 1.5759 \sin 2t)e^{0.0402t} \\ (-1.1927 \cos 2t + 0.4078 \sin 2t)e^{0.1598t} \end{bmatrix}. \quad (3.28)$$

The transformation given either by equation (3.11) or equation (3.12) also changes with normalization in such a way that equation (3.19) with  $\mathbf{h}(t)$  specified by equation (3.28) is generated. As explained earlier, canonical forms generated by different normalization schemes are regarded as equivalent.

### 3.2.2 Defective Systems

After a defective system has been converted into a decoupled system represented by equation (1.3), the canonical form (3.1) is obtained through application of transformation (3.8). In this case, the diagonal coefficient matrix  $\mathbf{B}$  is still given by equation (3.9), and the excitation has the form

$$\mathbf{h}(t) = \exp\left(\frac{1}{2}\mathbf{D}t\right) \mathbf{g}(t) = \exp\left(\frac{1}{2}\mathbf{D}t\right) \left\{ \left(\mathbf{D} + \mathbf{I}\frac{d}{dt}\right) \mathbf{G}_1(t)\mathbf{f}(t) + \mathbf{G}_2(t)\mathbf{f}(t) \right\}. \quad (3.29)$$

Combining equations (3.8) and (2.55) yields the configuration space transformation relating  $\mathbf{q}$  and  $\mathbf{x}$ :

$$\mathbf{q} = \left(\mathbf{T}_1(t) + \mathbf{T}_2(t)\frac{d}{dt}\right) \exp\left(-\frac{1}{2}\mathbf{D}t\right) \mathbf{x} - \mathbf{T}_2(t)\mathbf{G}_1(t)\mathbf{f}(t). \quad (3.30)$$

When equations (3.8) and (2.54) are combined, the transformation connecting equations (1.1) and (3.1) in the state space is obtained:

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \exp\left(\frac{1}{2}\mathbf{D}t\right) & \mathbf{0} \\ \frac{1}{2}\mathbf{D}\exp\left(\frac{1}{2}\mathbf{D}t\right) & \exp\left(\frac{1}{2}\mathbf{D}t\right) \end{bmatrix} \left\{ \mathbf{T}^{-1}(t) \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_1(t)\mathbf{f}(t) \end{bmatrix} \right\}. \quad (3.31)$$

Note that equations (3.29)-(3.31) hold for any defective system. If a defective system has all complex conjugate eigenvalues, then the matrices  $\mathbf{D}$  and  $\mathbf{\Omega}$  that characterize the canonical form (3.1) are as specified in equation (2.61), and the matrices  $\mathbf{G}_1(t)$ ,  $\mathbf{G}_2(t)$ ,  $\mathbf{T}_1(t)$ ,  $\mathbf{T}_2(t)$  and  $\mathbf{T}(t)$  in equations (3.29)-(3.31) are given by equations (2.62)-(2.68). Should this system be non-defective, then it is easy to verify that all formulae for transforming equation (1.1) into the canonical form (3.1) reduce to their non-defective counterparts.

#### 3.2.2.1 Illustrative Example: a Defective Passive System

A two-degree-of-freedom system is governed by

$$\ddot{\mathbf{q}} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 5 & -1 \\ -1 & 10 \end{bmatrix} \mathbf{q} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cos t, \quad (3.32)$$

with initial values  $\mathbf{q}(0) = \mathbf{0}$  and  $\dot{\mathbf{q}}(0) = \mathbf{0}$ . Solution of the quadratic eigenvalue problem (2.8) reveals that the system is defective with a repeated complex eigenvalue such that

$$\mathbf{J} = \begin{bmatrix} -1 + i\sqrt{6} & 0 \\ 0 & -1 + i\sqrt{6} \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} -i\sqrt{6}/2 & 5/2 \\ 1 & 0 \end{bmatrix} \quad (3.33)$$

and

$$\mathbf{\Lambda} = (-1 + i\sqrt{6}) \mathbf{I}, \quad \mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad (3.34)$$

The real and diagonal coefficients of the decoupled equation (1.3) are given by

$$\mathbf{D} = 2\mathbf{I}, \quad \mathbf{\Omega} = 7\mathbf{I}. \quad (3.35)$$

From equations (2.67) and (2.68),

$$\mathbf{G}_1 = \begin{bmatrix} 0 & -t/6 \\ 0 & 1/6 \end{bmatrix}, \quad \mathbf{G}_2 = \begin{bmatrix} -t/2 & t/6 + 5/6 \\ 1/2 & -1/6 \end{bmatrix}. \quad (3.36)$$

The canonical form (3.1) for system (3.32) is then specified by

$$\ddot{\mathbf{x}} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \mathbf{x} = \mathbf{h}(t), \quad (3.37)$$

where the excitation

$$\mathbf{h}(t) = \frac{e^t}{6} \begin{bmatrix} -8 \cos t - t(\cos t + 2 \sin t) \\ \cos t + 2 \sin t \end{bmatrix}. \quad (3.38)$$

The coordinates  $\mathbf{q}$  and  $\mathbf{x}$  are related in the configuration space by transformation (3.30):

$$\mathbf{q} = \left( \begin{bmatrix} 0 & 5e^{-t}/2 \\ e^{-t} & te^{-t} \end{bmatrix} + \begin{bmatrix} -e^{-t}/2 & -te^{-t}/2 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \right) \mathbf{x}. \quad (3.39)$$

Reduction of equation (3.32) to equation (3.37) is accomplished in state space by equation (3.31), for which

$$\begin{bmatrix} \exp(\frac{1}{2}\mathbf{D}t) & \mathbf{0} \\ \frac{1}{2}\mathbf{D} \exp(\frac{1}{2}\mathbf{D}t) & \exp(\frac{1}{2}\mathbf{D}t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} e^t, \quad (3.40)$$

$$\mathbf{T}(t) = \begin{bmatrix} -1/2 & 5/2 & -t/2 & -1/2 & -t/2 \\ 1 & t & 0 & 0 & 0 \\ 7/2 & 7t/2 & -1/2 & 1/2 & t/2 + 2 \\ 0 & 1 & 1 & t & 0 \end{bmatrix}, \quad (3.41)$$

and

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{G}_1(t)\mathbf{f}(t) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 0 \\ t \cos t \\ -\cos t \end{bmatrix}. \quad (3.42)$$

The initial values of equation (3.37) computed from (3.31) are  $\mathbf{x} = \mathbf{0}$  and  $\dot{\mathbf{x}} = [0 \quad -1/3]^T$ . As in the non-defective case, the canonical form generated is dependent on the normalization of eigenvectors. For example, if instead

$$\mathbf{V} = \begin{bmatrix} -0.1250 - 0.2041i & 0.5 - 0.2552i \\ 0.1667 - 0.1021i & 0.0680i \end{bmatrix}, \quad (3.43)$$

then the excitation  $\mathbf{h}(t)$  in equation (3.37) becomes

$$\mathbf{h}(t) = e^t \begin{bmatrix} -8.7438 \cos t + 2.5124 \sin t - t(3.4545 \cos t + 0.9091 \sin t) \\ 3.4545 \cos t + 0.9091 \sin t \end{bmatrix}. \quad (3.44)$$

The homogeneous part of equation (3.37) is unaffected by normalization because it is constructed only from the eigenvalues. The transformation given either by equation (3.30) or equation (3.31) changes with normalization in such a way that equation (3.37) with  $\mathbf{h}(t)$  specified by equation (3.44) is generated.

### 3.3 Conclusions

It has been shown that linear systems governed by equation (1.1) can be reduced to a canonical form specified by equation (3.1), a decoupled equation devoid of the velocity term and with the identity matrix as the coefficient of acceleration. While an exhaustive derivation has been provided only for non-defective systems with distinct eigenvalues, the reduction is applicable to both non-defective and defective linear systems possessing either symmetric or non-symmetric coefficient matrices. Major findings are summarized in the following statements.

1. All parameters required to construct the invertible transformation to convert equation (1.1) into equation (3.1) are obtained through the solution of the quadratic eigenvalue problem (2.8). For systems with distinct eigenvalues, the transformation is given either by equation (3.11) or equation (3.12), both of which are nonlinear.
2. For non-defective systems, different indexing and normalization schemes generate an equivalence class of canonical forms of the type defined by equation (3.1). If there are  $2n_r$  real eigenvalues of equation (2.8), then not more than  $N_r$  members of this equivalence class have different homogeneous parts, where  $N_r$  is given by equation (3.13). If all eigenvalues of equation (2.8) are complex, the canonical form (3.1) is unique up to the normalization of eigenvectors.
3. As an important by-product, a solution to the well-trodden problem of reducing a damped passive system to an undamped form has been provided.

Almost all linear systems are non-defective with distinct eigenvalues, and an emphasis has been placed on such systems. In the reduction of the equation of motion, the canonical form specified by equation (3.1) is the simplest representation of linear systems. Two examples have been supplied for illustration.

## Chapter 4

# Characterization of Free Motion of Passive Systems

A passive  $n$ -degree-of-freedom linear system is characterized by three symmetric and positive definite coefficient matrices. For SDOF passive systems (taking  $n = 1$ ), each coefficient is a positive number. For these systems, characterization of the free motion of a SDOF damped system is well understood [2–16]: the nature of oscillatory motion can be determined by inspection of the viscous damping ratio.

While a similar criterion for determining oscillatory behavior of the free motion of MDOF damped systems is desired, the situation here is less clear. This culminates in various criteria for determining the free response characteristics being reported in the literature. For example, a sufficient condition for non-oscillatory behavior is Duffin's overdamping condition [67]. However, this condition and others reported in [68–70] are rather difficult to verify and have not really found their ways into applications. Morzfeld et al. [39] introduced a viscous damping function that represents a direct extension of the classical damping ratio and is applicable to MDOF systems. The effect of viscous damping on the free motion is then determined by minimization and maximization of this viscous damping function. However, optimization of the viscous damping function may be problematic because the iterations can get trapped around local extrema in applications. Other approaches to this problem [40–44] rely upon simultaneous diagonalization of the coefficient matrices by linear coordinate transformations: these techniques apply to classically damped systems only and are not applicable in general.

The purpose is to study the free response characteristics of MDOF passive systems using phase synchronization. It is shown that the undamped, critically damped and overdamped degrees of freedom arise from pairing of eigenvalues upon decoupling. In addition, a system damping ratio can be constructed as a direct extension of the damping ratio for SDOF systems. This system damping ratio is a real number that depends on the system's coefficient matrices and it allows to determine whether oscillatory behavior is present or not.

The organization of the chapter is as follows. Phase synchronization is used to characterize the oscillatory behavior of each degree of freedom in section 4.1. This is followed

in section 4.2 where a system damping ratio is constructed from a direct extension of the damping ratio definition for a SDOF system. Several examples are supplied for illustration. A summary of findings is provided in section 4.3.

## 4.1 Characterization of Oscillatory Behavior by Eigenvalues

### 4.1.1 Background on SDOF Systems

Consider a damped SDOF system

$$m\ddot{q} + c\dot{q} + kq = 0. \quad (4.1)$$

The damping ratio is a non-negative number defined by the coefficients of the system:

$$\zeta = \frac{c}{2\sqrt{mk}}. \quad (4.2)$$

The SDOF system is underdamped if  $\zeta < 1$ , it is overdamped if  $\zeta > 1$  and it is critically damped if  $\zeta = 1$ . Oscillatory behavior is present whenever  $\zeta < 1$ .

The associated characteristic equation is

$$m\lambda^2 + c\lambda + k = 0, \quad (4.3)$$

whose roots are  $\lambda_1$  and  $\lambda_2$ . In terms of  $\zeta$ , the roots can be expressed as

$$\lambda_1, \lambda_2 = \frac{1}{2m} \left( -c \pm \sqrt{c^2 - 4mk} \right) = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}, \quad (4.4)$$

where  $\omega_n = \sqrt{k/m} > 0$  is the natural frequency. When the SDOF system is underdamped,  $\zeta < 1$ ; this happens when  $\lambda_1$  and  $\lambda_2$  are complex conjugates. If the system is overdamped,  $\zeta > 1$  and this means  $\lambda_1$  and  $\lambda_2$  are distinct negative real numbers. In the case it is critically damped,  $\zeta = 1$ , in which case  $\lambda_1 = \lambda_2 < 0$ .

The damping ratio can also be expressed in terms of the roots of the characteristic equation as

$$\zeta = \frac{c/m}{2\sqrt{k/m}} = -\frac{\lambda_1 + \lambda_2}{2} \frac{1}{\sqrt{\lambda_1\lambda_2}} = -\frac{\text{arithmetic mean of roots}}{\text{geometric mean of roots}}. \quad (4.5)$$

Here, the terms arithmetic and geometric means are used liberally because the roots may be complex. The inequality governing arithmetic and geometric means is valid only for positive numbers [71].

### 4.1.2 MDOF Systems

A passive system of the form (1.1) has an associated quadratic eigenvalue problem (2.8). The eigenvalue  $\lambda_j$  with eigenvector  $\mathbf{v}_j$  satisfies

$$\lambda_j = \frac{-\mathbf{v}_j^* \mathbf{C} \mathbf{v}_j \pm \sqrt{(\mathbf{v}_j^* \mathbf{C} \mathbf{v}_j)^2 - 4(\mathbf{v}_j^* \mathbf{M} \mathbf{v}_j)(\mathbf{v}_j^* \mathbf{K} \mathbf{v}_j)}}{2\mathbf{v}_j^* \mathbf{M} \mathbf{v}_j}, \quad (4.6)$$

where  $\mathbf{v}_j^*$  denotes the complex conjugate transpose of  $\mathbf{v}_j$ . Due to the positive definiteness of the coefficient matrices, any real eigenvalue is negative, while any complex conjugate pairs have negative real part. The number of complex and real eigenvalues of the quadratic eigenvalue problem (2.8) is fixed for any given system.

Any linear passive system (1.1) can always be decoupled in real space into (1.3) by solving the quadratic eigenvalue problem (2.8). Phase synchronization generates all possible decoupled forms into which a system can be transformed in real space [24, 25]. The real and diagonal coefficient matrices of the decoupled system (1.3), given by (2.17), are clearly positive definite if  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$  are positive definite. Since complex conjugate eigensolutions must be paired, the decoupled system is unique when all eigenvalues are complex [23, 25]. However, real eigensolutions can be paired in different ways, leading to different decoupled forms [24, 25].

Suppose an  $n$ -degree-of-freedom is decoupled by phase synchronization. The degree of freedom  $p_j$  ( $j = 1, 2, \dots, n$ ) satisfies a second order differential equation of the form

$$\ddot{p}_j - (\lambda_j + \lambda_{n+j})\dot{p}_j + (\lambda_j \lambda_{n+j})p_j = 0, \quad (4.7)$$

where  $\lambda_{n+j} = \bar{\lambda}_j$  whenever  $\lambda_j$  is complex. As in the case for SDOF systems (4.5), denote the damping ratio of the  $j$ th scalar equation of the decoupled system (1.3) by

$$\zeta_{pj} = -\frac{\lambda_j + \lambda_{n+j}}{2} \frac{1}{\sqrt{\lambda_j \lambda_{n+j}}}. \quad (4.8)$$

When  $\lambda_j$  is complex,  $\lambda_{n+j} = \bar{\lambda}_j$  and  $\zeta_{pj} < 1$ . This means each pair of complex conjugate eigenvalues produces an underdamped scalar equation, or underdamped degree of freedom in the decoupled system. When  $\lambda_j$  and  $\lambda_{n+j}$  are real eigenvalues, they produce an overdamped degree of freedom when  $\lambda_j \neq \lambda_{n+j}$  or a critically damped degree of freedom when  $\lambda_j = \lambda_{n+j}$ . This leads to the conclusion in Theorem 4.1:

**Theorem 4.1.** *If there are  $2n_c$  complex conjugate eigenvalues and  $2n_r = 2(n - n_c)$  real eigenvalues, then  $n_c$  degrees of freedom are underdamped and  $n_r$  degrees of freedom are overdamped or critically damped.*

Because of coupling, the system response is oscillatory if at least one scalar equation in the decoupled system is underdamped. This is an exact and complete characterization but knowledge of the eigenvalues is required.

#### 4.1.2.1 Illustrative Example: Characterization by Eigenvalues

Consider the three-degree-of-freedom system

$$\ddot{\mathbf{q}} + \begin{bmatrix} 12 & 10 & 0 \\ 10 & 12 & -1 \\ 0 & -1 & 2 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 12 & 11 & 6 \\ 11 & 12 & 5 \\ 6 & 5 & 30 \end{bmatrix} \mathbf{q} = \mathbf{0}. \quad (4.9)$$

Solution of the quadratic eigenvalue problem (2.8) reveals that the spectrum consists of:

- a complex conjugate pair  $\lambda_1 = -1.0330 + 5.4034i$ ,  $\lambda_4 = \bar{\lambda}_1$  with eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -0.2441 - 0.3290i \\ 1.8187 - 6.714i \end{bmatrix}, \quad \mathbf{v}_4 = \bar{\mathbf{v}}_1. \quad (4.10)$$

- two simple real eigenvalues,  $\lambda_2 = -20.9597$  and  $\lambda_6 = -0.9743$  with eigenvectors

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1.0038 \\ -0.0750 \end{bmatrix}, \quad \mathbf{v}_6 = \begin{bmatrix} 1 \\ -0.7817 \\ -0.0459 \end{bmatrix}. \quad (4.11)$$

- a defective real eigenvalue repeated two times:  $\lambda_3 = -1$ . This eigenvalue has unit geometric multiplicity and its eigenvector is

$$\mathbf{v}_3^1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}. \quad (4.12)$$

The associated generalized eigenvector  $\mathbf{v}_3^2$ , given by (2.44) and (2.45), is

$$\mathbf{v}_3^2 = \begin{bmatrix} 1 \\ -1.8571 \\ 0.1429 \end{bmatrix}. \quad (4.13)$$

Note that each eigenvector was normalized so that its first element is 1. Conditions of theorem 2.1 are met, so the indexing of eigenvalues

$$\begin{aligned} \mathbf{\Lambda}_1 &= \begin{bmatrix} -1.0330 + 5.4034i & 0 & 0 \\ 0 & -20.9597 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\ \mathbf{\Lambda}_2 &= \begin{bmatrix} -1.0330 - 5.4034i & 0 & 0 \\ 0 & -0.9743 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned} \quad (4.14)$$



originate the decoupled system

$$\ddot{\mathbf{p}} + \begin{bmatrix} 2.0660 & 0 & 0 \\ 0 & 21.9340 & 0 \\ 0 & 0 & 2 \end{bmatrix} \dot{\mathbf{p}} + \begin{bmatrix} 30.2638 & 0 & 0 \\ 0 & 20.4210 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p} = \mathbf{0}, \quad (4.15)$$

where the complex eigenvalues produce an underdamped degree of freedom with  $\zeta_{p1} = 0.1878$  while the distinct real eigenvalues originate an overdamped degree of freedom with  $\zeta_{p2} = 2.4269$ . The defective real eigenvalue forms a critically damped degree of freedom with  $\zeta_{p3} = 1$ . To compute the decoupling transformation, let

$$\mathbf{J}_q = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\lambda}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{bmatrix}, \quad \mathbf{V}_q = [\mathbf{v}_1 \quad \bar{\mathbf{v}}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_6 \quad \mathbf{v}_3^1 \quad \mathbf{v}_3^2]. \quad (4.16)$$

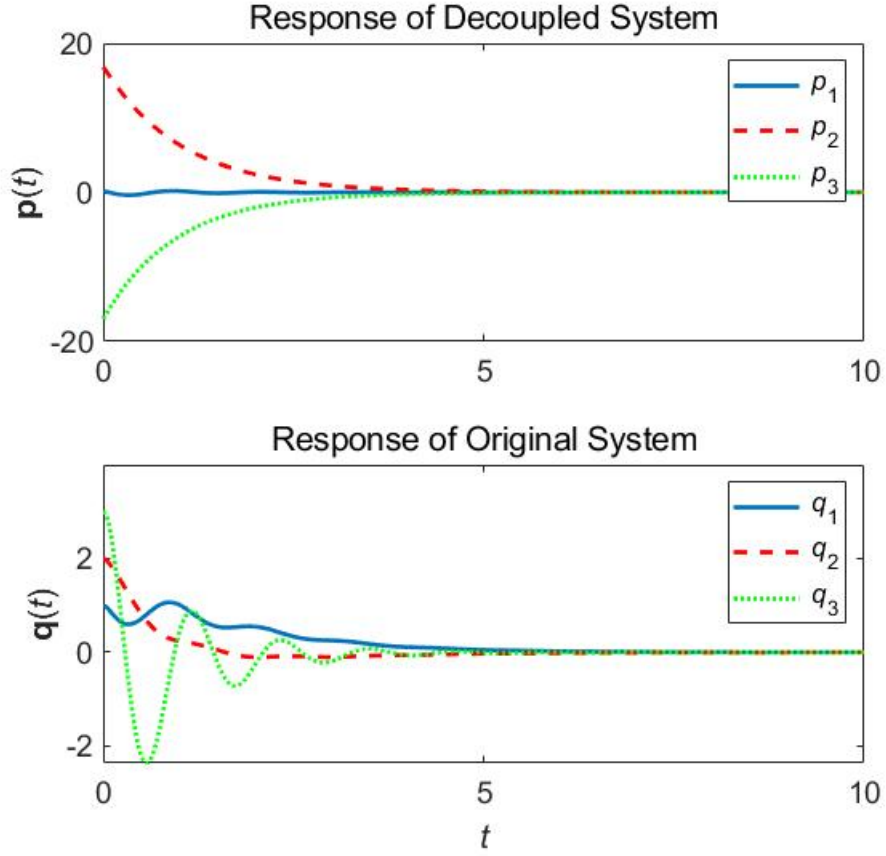
For isospectral decoupling, define the matrix

$$\mathbf{V}_p = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \quad (4.17)$$

The decoupling transformation is then given by (2.46)

$$\begin{aligned} \mathbf{q}(t) &= \left( \mathbf{T}_1 + \mathbf{T}_2 \frac{d}{dt} \right) \mathbf{p}(t) \\ &= \begin{bmatrix} 1 & 1 & 2 \\ -0.3070 & -0.8687 & -2.8571 \\ 0.5357 & -0.0444 & 0.1429 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 & 1 \\ -0.0609 & -0.0893 & -1.8571 \\ -1.2421 & 0.0015 & 0.1429 \end{bmatrix} \begin{bmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \\ \dot{p}_3(t) \end{bmatrix}. \end{aligned} \quad (4.18)$$

The response of coupled and decoupled systems are shown in figure 9.



**Fig. 9** The response  $\mathbf{q}(t)$  of the coupled defective system and the response  $\mathbf{p}(t)$  of the decoupled system: complex conjugate eigenvalues originate an under-damped degree of freedom with  $\zeta_{p1} = 0.1878$ , distinct real eigenvalues originate an overdamped degree of freedom with  $\zeta_{p2} = 2.4269$  and repeated real eigenvalues originate a critically damped degree of freedom with  $\zeta_{p3} = 1$ . The initial conditions used for the simulation are  $\mathbf{q}(0) = [1 \ 2 \ 3]^T$  and  $\dot{\mathbf{q}}(0) = \mathbf{0}$ .

Now, one can enforce just the preservation of eigenvalues. In this case, the eigenvalues are paired in accordance with

$$\begin{aligned} \mathbf{\Lambda}_1 &= \begin{bmatrix} -1.0330 + 5.4034i & 0 & 0 \\ 0 & -20.9597 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\ \mathbf{\Lambda}_2 &= \begin{bmatrix} -1.0330 - 5.4034i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -0.9743 \end{bmatrix} \end{aligned} \quad (4.19)$$

and the decoupled system becomes

$$\ddot{\mathbf{p}} + \begin{bmatrix} 2.0660 & 0 & 0 \\ 0 & 21.9597 & 0 \\ 0 & 0 & 1.9743 \end{bmatrix} \dot{\mathbf{p}} + \begin{bmatrix} 30.2638 & 0 & 0 \\ 0 & 20.9597 & 0 \\ 0 & 0 & 0.9743 \end{bmatrix} \mathbf{p} = \mathbf{0}. \quad (4.20)$$

The complex eigenvalues always produce an underdamped degree of freedom, but the pairing of real eigenvalues originate two overdamped degree of freedom with  $\zeta_{p2} = 2.3983$  and  $\zeta_{p3} = 1.0001$ . To obtain the decoupling transformation, let

$$\mathbf{J}_q = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\lambda}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{bmatrix}, \quad \mathbf{V}_p = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}. \quad (4.21)$$

The decoupling transformation is then given by

$$\begin{aligned} \mathbf{q}(t) &= \left( \mathbf{T}_1 + \mathbf{T}_2 \frac{d}{dt} \right) \mathbf{p}(t) \\ &= \begin{bmatrix} 1 & 1 & 1 - 37.8524t \\ -0.3070 & -1.1004 & 37.8524t + 39.9263 \\ 0.5357 & 0.0038 & -7.1892 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 & -38.8524t \\ -0.0609 & -1.1004 & 38.8524t + 41.7835 \\ -1.2421 & 0.0038 & -7.3320 \end{bmatrix} \begin{bmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \\ \dot{p}_3(t) \end{bmatrix}. \end{aligned} \quad (4.22)$$

## 4.2 Characterization of Oscillatory Behavior by Damping Ratio

In the literature, criteria involving functionals were given to determine if the system response is oscillatory. For example, a system is said to be overdamped when the classical overdamping condition

$$D(\mathbf{z}) = (\mathbf{z}^* \mathbf{C} \mathbf{z})^2 - 4 (\mathbf{z}^* \mathbf{M} \mathbf{z}) (\mathbf{z}^* \mathbf{K} \mathbf{z}) > 0 \quad (4.23)$$

is satisfied for all  $\mathbf{z} \in \mathbb{C}^n$ . When the overdamping condition is satisfied, all eigenvalues are negative and the free response is exponentially decaying [61, 64]. However, the overdamping condition is only sufficient and not necessary for overdamping, as demonstrated in the following example. Consider a system (1.1) with

$$\mathbf{M} = \mathbf{I}, \quad \mathbf{C} = \begin{bmatrix} 19 & 9 \\ 9 & 19 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 91 & 70 \\ 70 & 91 \end{bmatrix}. \quad (4.24)$$

The eigenvalues of (2.8) can be arranged into a primary-secondary scheme as

$$\mathbf{\Lambda}_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -19.9161 & 0 \\ 0 & -8.0839 \end{bmatrix}, \quad \mathbf{\Lambda}_2 = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -3 \end{bmatrix}. \quad (4.25)$$

In this system, all eigenvalues are negative and distinct; therefore all DOF are overdamped. The input  $\mathbf{z} = [0 \ 1]^T$  gives

$$D([0 \ 1]^T) = -3. \quad (4.26)$$

The overdamping condition is not satisfied, even though each degree of freedom is overdamped, as also seen in figure 10. Damping criteria involving functionals are not convenient to use. A criterion that utilizes a real number to determine if the system response is oscillatory is preferred.

When an  $n$ -degree-of-freedom is decoupled by phase synchronization, the damping ratio of the  $j$ th degree of freedom is given by (4.8). Since different schemes can be used to pair real eigensolutions, parameters to characterize damping for the coupled system (1.1) such as

$$\zeta_p = \zeta_{p1} \zeta_{p2} \cdots \zeta_{pn}, \quad (4.27)$$

$$\zeta_{\max} = \max_{1 \leq j \leq n} \zeta_{pj}, \quad (4.28)$$

or

$$\zeta_{\min} = \min_{1 \leq j \leq n} \zeta_{pj} \quad (4.29)$$

are not fixed and their values are dependent on the chosen pairing scheme. Moreover, the computation of these parameters, or even of each of  $\zeta_{pj}$  ( $j = 1, 2, \dots, n$ ) depend on the solution of the quadratic eigenvalue problem (2.8). A parameter that depends only on the given system's parameters, such as the coefficient matrices, is desired.

### 4.2.1 Damping Ratio for Systems and Oscillatory Behavior

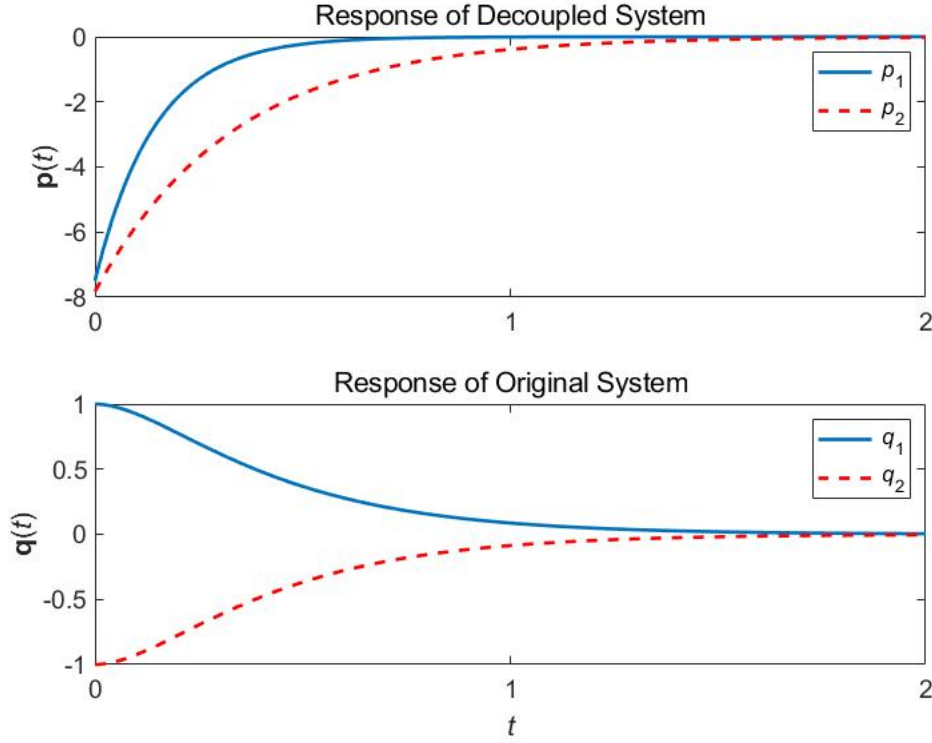
For an  $n$ -degree-of-freedom system (1.1), define the system damping ratio by

$$\zeta = \frac{\text{tr}(\mathbf{M}^{-1}\mathbf{C})}{2n} \frac{1}{\sqrt[2n]{\det(\mathbf{M}^{-1}\mathbf{K})}}. \quad (4.30)$$

Since  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$  are positive definite, the system damping ratio  $\zeta$  is a non-negative number [31, pp. 430, 486]. Furthermore,  $\zeta$  is, by construction, independent of the pairing scheme for real eigensolutions since it only depends on the coefficient matrices. In addition, when  $n = 1$ , each coefficient matrix is just a positive real number and  $\zeta$  in (4.30) reduces to the well-known damping ratio for SDOF systems (4.5).

It will be proved that the system response in free vibration is oscillatory if  $\zeta < 1$ . The  $2n$  eigenvalues  $\lambda_j$  of (2.8) are the same as those of the  $2n \times 2n$  state companion matrix given by [4, pp. 346-355]

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}. \quad (4.31)$$



**Fig. 10** Response  $\mathbf{q}(t)$  of system (1.1) with coefficients (4.24) and the response  $\mathbf{p}(t)$  of the decoupled system obtained by phase synchronization. The initial conditions  $\mathbf{q}(0) = [1 \ -1]^T$  and  $\dot{\mathbf{q}}(0) = \mathbf{0}$  were used in the simulation. Even though each degree of freedom is overdamped, the overdamping condition is not satisfied.

The trace and determinant of  $\mathbf{A}$  can both be related to its eigenvalues [31, p. 51]. As a consequence [72],

$$\prod_{j=1}^{2n} \lambda_j = \det(\mathbf{A}) = \det(\mathbf{M}^{-1}\mathbf{K}) = \frac{\det(\mathbf{K})}{\det(\mathbf{M})}, \quad (4.32)$$

and

$$\sum_{j=1}^{2n} \lambda_j = \text{tr}(\mathbf{A}) = -\text{tr}(\mathbf{M}^{-1}\mathbf{C}). \quad (4.33)$$

It follows that

$$\zeta = -\frac{\sum_{j=1}^{2n} \lambda_j}{2n} \frac{1}{\left(\prod_{j=1}^{2n} \lambda_j\right)^{1/2n}} = -\frac{\text{arithmetic mean of eigenvalues}}{\text{geometric mean of eigenvalues}}. \quad (4.34)$$

Again, the terms arithmetic and geometric mean are used loosely since each  $\lambda_j$  ( $j = 1, 2, \dots, 2n$ ) is either a real negative number or complex with negative real part. The arithmetic mean of all  $-\lambda_j$  can be written as

$$-\frac{1}{2n} \sum_{j=1}^{2n} \lambda_j = \frac{1}{n} \sum_{j=1}^n \left( \frac{-\lambda_j - \lambda_{n+j}}{2} \right), \quad (4.35)$$

where each pair  $\{\lambda_j, \lambda_{n+j}\}$  ( $j = 1, 2, \dots, n$ ) is the same pairing utilized in phase synchronization. In phase synchronization, complex conjugate eigensolutions must be paired. Thus

$$-\lambda_j - \lambda_{n+j} = 2|\operatorname{Re}(\lambda_j)| \quad (4.36)$$

is always the absolute value of the real part of a complex eigenvalue  $\lambda_j$ . Real eigenvalues are negative, so each pairing of real eigenvalues in (4.35) results in a positive real number. Therefore, the pairing results in the arithmetic mean of  $n$  positive numbers, and the inequality governing arithmetic and geometric means is applicable [71]:

$$\frac{1}{n} \sum_{j=1}^{2n} \left( \frac{-\lambda_j - \lambda_{n+j}}{2} \right) \geq \left[ \prod_{j=1}^n \left( \frac{-\lambda_j - \lambda_{n+j}}{2} \right) \right]^{1/n}. \quad (4.37)$$

It follows that

$$\zeta \geq \left[ \prod_{j=1}^n \left( \frac{-\lambda_j - \lambda_{n+j}}{2} \right) \right]^{1/n} \frac{1}{\left( \prod_{j=1}^{2n} \lambda_j \right)^{1/2n}} = \left[ \prod_{j=1}^n \left( \frac{-\lambda_j - \lambda_{n+j}}{2\sqrt{\lambda_j \lambda_{n+j}}} \right) \right]^{1/n} \quad (4.38)$$

This means

$$\zeta \geq (\zeta_{p1} \zeta_{p2} \dots \zeta_{pn})^{1/n}, \quad (4.39)$$

where  $\zeta_{pj}$  ( $j = 1, \dots, n$ ) is given by (4.8). If  $\zeta < 1$ , at least one  $\zeta_{pj} < 1$ . Thus, the following theorem holds:

**Theorem 4.2.** *The free response of a MDOF damped system is oscillatory if the damping ratio  $0 < \zeta < 1$ .*

In other words, at least one degree of freedom is underdamped. Again, the system response is oscillatory if at least one scalar equation in the decoupled system is underdamped due to coupling. It is practical to use Theorem 4.2 first to check for oscillatory behavior and, if necessary, Theorem 4.1 can then be used for a complete characterization.

#### 4.2.1.1 Illustrative Example: Prediction of Oscillatory Behavior

As an example, let

$$\mathbf{M} = \mathbf{I}, \quad \mathbf{C} = \begin{bmatrix} 3.4 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & -4 \\ -4 & 26 \end{bmatrix}. \quad (4.40)$$

The system damping ratio is

$$\zeta = \frac{\text{tr}(\mathbf{C})}{4} \frac{1}{\sqrt[4]{\det(\mathbf{K})}} = 0.6532, \quad (4.41)$$

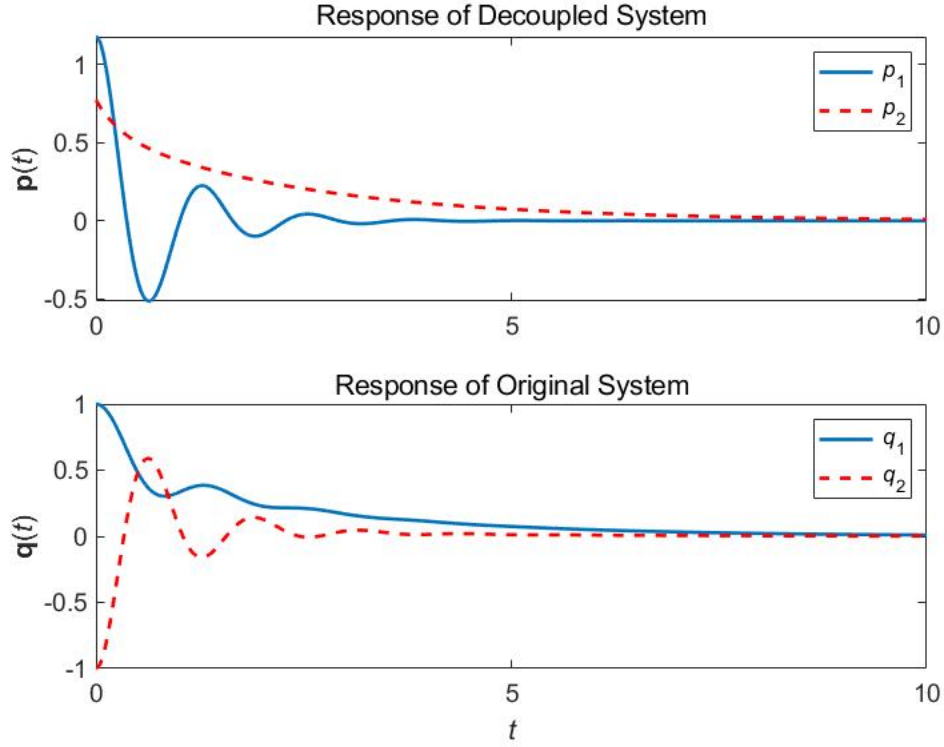
so at least one degree of freedom is underdamped and oscillatory behavior is present by Theorem 4.2. To see that, arrange the solutions of the quadratic eigenvalue problem (2.8) in primary-secondary scheme as

$$\mathbf{\Lambda}_1 = \begin{bmatrix} -1.2957 + 4.9258i & 0 \\ 0 & -3.4005 \end{bmatrix}, \quad \mathbf{\Lambda}_2 = \begin{bmatrix} -1.2957 - 4.9258i & 0 \\ 0 & -0.4081 \end{bmatrix}. \quad (4.42)$$

By Theorem 4.1, this system has one overdamped and one underdamped degree of freedom. The decoupled system is given by

$$\ddot{\mathbf{p}} + \begin{bmatrix} 2.5914 & 0 \\ 0 & 3.8086 \end{bmatrix} \dot{\mathbf{p}} + \begin{bmatrix} 25.9427 & 0 \\ 0 & 1.3877 \end{bmatrix} \mathbf{p} = \mathbf{0}. \quad (4.43)$$

The system has an underdamped and an overdamped degree of freedom with  $\zeta_{p1} = 0.2544$  and  $\zeta_{p2} = 1.6165$ , respectively. As a visual aid, the responses of the original and coupled systems are shown in figure 11.



**Fig. 11** Response  $\mathbf{q}(t)$  of system (1.1) with coefficients (4.40) and the response  $\mathbf{p}(t)$  of the decoupled system obtained by phase synchronization. The initial conditions  $\mathbf{q}(0) = [1 \ -1]^T$  and  $\dot{\mathbf{q}}(0) = \mathbf{0}$  were used in the simulation. Since  $\zeta < 1$ , at least one degree of freedom is underdamped and the response  $\mathbf{q}(t)$  has oscillatory behavior.

#### 4.2.1.2 Illustrative Example: Upper Bound

Among different damped systems,  $\zeta$  is an upper bound of  $(\zeta_{p1}\zeta_{p2}\dots\zeta_{pn})^{1/n}$ . The upper bound can be attained in many systems. As an example, let

$$\mathbf{M} = \mathbf{I}, \quad \mathbf{C} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & 0 \\ 0 & 0.75 \end{bmatrix}. \quad (4.44)$$

The system has an underdamped and an overdamped degree of freedom with  $\zeta_{p1} = 0.7071$  and  $\zeta_{p2} = 1.1547$ , respectively. It can be checked that  $\zeta = (\zeta_{p1}\zeta_{p2})^{1/2} = 0.9036$ .

#### 4.2.1.3 Illustrative Example: Lower Bound

A lower bound of  $(\zeta_{p1}\zeta_{p2}\dots\zeta_{pn})^{1/n}$  is zero and this is the case if any degree of freedom is undamped. Since  $\mathbf{C}$  is positive definite, this lower bound cannot be attained. However,



systems can be constructed for which  $(\zeta_{p1}\zeta_{p2}\dots\zeta_{pn})^{1/n}$  is very small. As an example, let

$$\mathbf{M} = \mathbf{I}, \quad \mathbf{C} = \begin{bmatrix} 20 & -14 \\ -14 & 10 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 32 & 24 \\ 24 & 38 \end{bmatrix}. \quad (4.45)$$

Solution of the quadratic eigenvalue problem (2.8) yields

$$\mathbf{\Lambda}_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -0.0745 + 7.657i & 0 \\ 0 & -29.4808 \end{bmatrix} \quad (4.46)$$

and

$$\mathbf{\Lambda}_2 = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} -0.0745 - 7.657i & 0 \\ 0 & -0.3702 \end{bmatrix}. \quad (4.47)$$

The system has an underdamped and an overdamped degree of freedom with  $\zeta_{p1} = 0.0097$  and  $\zeta_{p2} = 4.5178$ , respectively. It is found that  $(\zeta_{p1}\zeta_{p2})^{1/2} = 0.0440$  while  $\zeta = 1.4911$ .

### 4.2.2 Invariance Under Equivalence Transformation

The system damping ratio (4.30) is invariant under any linear coordinate transformation, such as classical modal analysis. Suppose the system (1.1) is transformed by an equivalence transformation into

$$\mathbf{VMU}\ddot{\mathbf{p}} + \mathbf{VCU}\dot{\mathbf{p}} + \mathbf{VKU}\mathbf{p} = \mathbf{0}, \quad (4.48)$$

where  $\mathbf{V}$  and  $\mathbf{U}$  are invertible  $n \times n$  matrices. If  $\mathbf{V} = \mathbf{U}^{-1}$ , the equivalence transformation is a similarity transformation. If  $\mathbf{V} = \mathbf{U}^T$ , the equivalence transformation is a modal transformation.

Since  $\text{tr}(\mathbf{U}^{-1}\mathbf{CU}) = \text{tr}(\mathbf{C})$  and  $\det(\mathbf{U}^{-1}\mathbf{KU}) = \det(\mathbf{K})$  [31, p. 59], the parameter  $\zeta_p$  of the transformed system is

$$\begin{aligned} \zeta_p &= \frac{\text{tr}\{(\mathbf{VMU})^{-1}\mathbf{VCU}\}}{2n} \frac{1}{\sqrt[2n]{\det\{(\mathbf{VMU})^{-1}\mathbf{VKU}\}}} \\ &= \frac{\text{tr}(\mathbf{U}^{-1}\mathbf{M}^{-1}\mathbf{CU})}{2n} \frac{1}{\sqrt[2n]{\det(\mathbf{U}^{-1}\mathbf{M}^{-1}\mathbf{KU})}} \\ &= \frac{\text{tr}(\mathbf{M}^{-1}\mathbf{C})}{2n} \frac{1}{\sqrt[2n]{\det(\mathbf{M}^{-1}\mathbf{K})}} = \zeta. \end{aligned} \quad (4.49)$$

Thus, the following theorem was proved:

**Theorem 4.3.** *The system damping ratio  $\zeta$  given by (4.30) is invariant under equivalence transformation. Specifically,  $\zeta$  is invariant under modal transformations.*

### 4.2.3 Damping Ratio Exceeding One

For SDOF systems,  $\zeta > 1$  indicates overdamping. For MDOF systems, this result is no longer valid. If all degrees of freedom are overdamped,

$$\zeta_{pj} > 1 \quad (4.50)$$

for all  $j$  ( $j = 1, \dots, n$ ). Because  $\zeta \geq (\zeta_{p1}\zeta_{p2}\dots\zeta_{pn})^{1/n}$ , this clearly means  $\zeta > 1$ . However, it is not necessary for an overdamped degree of freedom to occur when  $\zeta > 1$ . In fact, all degrees of freedom may be underdamped:

**Theorem 4.4.** *If all degrees of freedom are overdamped, then  $\zeta > 1$ . The converse is not always true.*

#### 4.2.3.1 Illustrative Example: Underdamped Degrees of Freedom

As an example, let

$$\ddot{\mathbf{q}} + \begin{bmatrix} 10 & -4 \\ -4 & 10 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 31 & -19 \\ -19 & 31 \end{bmatrix} \mathbf{q} = \mathbf{0}. \quad (4.51)$$

This system is classically damped. Using modal analysis, the equation of motion is decoupled into

$$\ddot{\mathbf{p}} + \begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix} \dot{\mathbf{p}} + \begin{bmatrix} 50 & 0 \\ 0 & 12 \end{bmatrix} \mathbf{p} = \mathbf{0}. \quad (4.52)$$

All degrees of freedom are underdamped with damping ratios given by

$$\zeta_{p1} = \frac{14}{2\sqrt{50}} = 0.9899, \quad \zeta_{p2} = \frac{6}{2\sqrt{12}} = 0.8660. \quad (4.53)$$

On the other hand, the system damping ratio is

$$\zeta = \frac{\text{tr}(\mathbf{C})}{4} \frac{1}{\sqrt[4]{\det(\mathbf{K})}} = 1.0103. \quad (4.54)$$

As additional verification, the system eigenvalues are  $\lambda_1 = -7 + i$  and  $\lambda_2 = -7 + i\sqrt{3}$ , which also produces

$$\zeta = -\frac{\lambda_1 + \lambda_2 + \bar{\lambda}_1 + \bar{\lambda}_2}{4} \frac{1}{\sqrt[4]{\lambda_1\lambda_2\bar{\lambda}_1\bar{\lambda}_2}} = 1.0103. \quad (4.55)$$

#### 4.2.3.2 Illustrative Example: Critically Damped Degree of Freedom

If the system is defective, there may only be critically damped degrees of freedom and not any overdamped degree of freedom when  $\zeta > 1$ . Consider a system with eigenvalues -2, -2, -1, -1 so that the system damping ratio is  $\zeta = 1.0607$ . If the system is non-defective,

the eigenvalues -2 and -1 can be paired twice to produce an isospectral decoupled system with two overdamped degree of freedom and  $\zeta_{p1} = \zeta_{p2} = 1.0607$ . If the system is defective, then -2 is paired with -2 and -1 is paired with -1 to produce a decoupled system with two critically damped degree of freedom and  $\zeta_{p1} = \zeta_{p2} = 1$ . This example also demonstrates that  $\zeta_{p1}, \zeta_{p2}, \dots, \zeta_{pn}$  and its value depends on the pairing scheme used.

#### 4.2.4 Damping Ratio Equal to One

When  $\zeta = 1$ , there are two mutually exclusive possibilities. First, all  $\lambda_j$  ( $j = 1, 2, \dots, 2n$ ) are real. Then  $\zeta$  is a ratio of arithmetic mean to geometric mean of  $2n$  real numbers, and  $\zeta = 1$  if and only if  $\lambda_j$  are all equal. In this case, all degrees of freedom are identically and critically damped. Second, some  $\lambda_j$  are complex. Complex conjugate eigenvalues combine to decrease  $\zeta$  but real and unequal eigenvalues combine to increase  $\zeta$ . In general, underdamped, overdamped, and critically damped degrees of freedom can occur together when  $\zeta = 1$ .

##### 4.2.4.1 Illustrative Example: Underdamped and Overdamped Degrees of Freedom

As an example, let

$$\mathbf{M} = \mathbf{I}, \quad \mathbf{C} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}. \quad (4.56)$$

Solution of the quadratic eigenvalue problem (2.8) yields

$$\mathbf{\Lambda}_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1.2608 + 1.8669i & 0 \\ 0 & -1.3303 \end{bmatrix}, \quad (4.57)$$

and

$$\mathbf{\Lambda}_2 = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} -1.2608 - 1.8669i & 0 \\ 0 & -0.1481 \end{bmatrix}. \quad (4.58)$$

The system has an underdamped and an overdamped degree of freedom with  $\zeta_{p1} = 0.5597$  and  $\zeta_{p2} = 1.6653$ , respectively, even though  $\zeta = 1$ .

### 4.3 Conclusions

Determining the response characteristics of a damped MDOF linear system in free motion is important in analysis and design. Two methods were developed to determine oscillatory behavior of damped MDOF linear systems. Major findings are summarized in the following statements.

1. The response characteristics of a damped MDOF linear system can be completely characterized by the spectrum of the system. The  $2n_c$  complex conjugate eigenvalues

form  $n_c$  underdamped degrees of freedom. Distinct pairs of real eigenvalues generate overdamped degrees of freedom, while pairs of equal real eigenvalues produce critically damped ones.

2. A damping ratio for MDOF systems was constructed in (4.30). This damping ratio is a direct extension of the damping ratio for SDOF systems in (4.5). The damping ratio for systems (4.30) predicts oscillatory behavior whenever  $0 < \zeta < 1$ . If all degrees of freedom are overdamped, then  $\zeta > 1$ .
3. The damping ratio  $\zeta$  is preserved under any time-invariant linear transformations, including modal transformations.

While the system damping ratio indicates oscillatory behavior if  $0 < \zeta < 1$ , no information can be deduced for the cases when  $\zeta \geq 1$ . That might indicate a limitation when characterizing the response of a large number of equations by a single number.

## Chapter 5

# The Inverse Problem of Lagrangian Dynamics

The inverse problem of linear Lagrangian dynamics is concerned with finding a scalar function such that the associated Euler-Lagrange equations are equivalent to the assigned equations of motion. This scalar function is termed a Lagrangian.

Darboux [45] demonstrated the existence of Lagrangians for SDOF systems. Leitmann obtained Lagrangians associated with nonpotential forces for which a variational principle exists [46]. Subsequently, Udwadia et al. [47] derived the Lagrangians connected with general nonpotential forces. He [48] used the semi-inverse method to derive Lagrangians of the Korteweg-de Vries and Schrödinger equations. Musielak et al. [49] derived Lagrangians of nonlinear SDOF systems with variable coefficients and presented methods to obtain standard and nonstandard Lagrangians of SDOF systems [50]. A Lagrangian is referred to as standard (or natural) if it can be expressed as the difference between kinetic and potential energy terms; otherwise, the Lagrangian is termed nonstandard (or non-natural). These and other earlier works [51–53] have addressed the inverse problem for SDOF systems.

Solution of the inverse problem for MDOF systems poses a greater challenge because the equations of motion are usually coupled; it is thus not permissible to focus on individual component equations [2–16]. General conditions for the existence of Lagrangians are provided by the so-called Helmholtz conditions [54, 55], an assessment of which requires the solution of certain partial differential equations. Udwadia and Cho [56] obtained Lagrangians for a class of SDOF and MDOF linear systems by invoking the Helmholtz conditions. In general, the Helmholtz conditions offer little assistance in the solution of the inverse problem for MDOF systems. Douglas [57] and Crampin et al. [58] addressed the inverse problem for two degree-of-freedom systems using Riquier theory with an exhaustive case-by-case examination. Recently, Udwadia [29] obtained Lagrangians for classically damped linear systems using modal analysis. However, damped linear systems are generally not amenable to modal analysis [18].

A comprehensive study is reported herein for the evaluation of Lagrangian functions. It will be demonstrated that system decoupling, used successfully by Udwadia [29] for classically

damped systems, cannot be extended to obtain Lagrangians for general linear systems. It will also be shown that many coupled systems do not admit Lagrangian functions, but a scalar function that plays the role of a Lagrangian function, termed generalized Lagrangian, can be found for every linear system. Explicit conditions when a generalized Lagrangian coincide with a Lagrangian are discussed.

The organization of this chapter is as follows: The inverse problem of linear Lagrangian dynamics is formulated in section 5.1, and solutions for SDOF and classically damped MDOF linear systems are reviewed. The effect of decoupling transformations on the Euler-Lagrange equations is examined in section 5.2, where generalized Lagrangian functions are determined. Defective linear systems are explored in section 5.3, and the existence of Lagrangian functions for coupled linear systems is addressed in section 5.4. Finally, a summary of findings is provided in section 5.5. Five examples are supplied throughout the chapter for illustration.

## 5.1 Problem Statement

Because  $\mathbf{M}$  is assumed nonsingular, it becomes convenient to take  $\mathbf{M} = \mathbf{I}$  to streamline the presentation. In addition, only the homogeneous system will be considered, so rewrite (1.1) as

$$\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}, \quad (5.1)$$

where  $\mathbf{C}$  and  $\mathbf{K}$  are arbitrary, real  $n \times n$  matrices. Define the derivative of a multivariate scalar function  $F$  with respect to an  $n$ -dimensional vector such as  $\mathbf{q}$  in equation (1.2) by

$$\frac{\partial F}{\partial \mathbf{q}} = \left[ \frac{\partial F}{\partial q_1} \quad \frac{\partial F}{\partial q_2} \quad \cdots \quad \frac{\partial F}{\partial q_n} \right]^T. \quad (5.2)$$

Concisely speaking, the inverse problem of linear Lagrangian dynamics amounts to finding a scalar function  $L(\mathbf{q}, \dot{\mathbf{q}}, t)$  that satisfies the corresponding Euler-Lagrange equation:

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{\mathbf{q}}} \right] - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t) (\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q}) = \mathbf{0}. \quad (5.3)$$

where  $\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t)$  is a nonsingular  $n \times n$  matrix multiplier. A general solution to the inverse problem has never been reported in the open literature. However, Lagrangian functions have already been determined for SDOF and classically damped MDOF linear systems. These solutions are now summarized.

### 5.1.1 Lagrangians for SDOF Systems

A linear SDOF system of the form

$$\ddot{p} + d\dot{p} + bp = 0, \quad (5.4)$$

where  $d$  and  $b$  are constants, admits the Lagrangian function [29]

$$L(p, \dot{p}, t) = \frac{1}{2} \left( \dot{p}^2 + d\dot{p}p + \frac{d^2}{2}p^2 \right) e^{dt} - \frac{b}{2}p^2 e^{dt} \quad (5.5)$$

and, alternatively, a more compact Lagrangian function

$$L(p, \dot{p}, t) = \frac{1}{2}\dot{p}^2 e^{dt} - \frac{b}{2}p^2 e^{dt}. \quad (5.6)$$

As a direct verification, substitute either (5.5) or (5.6) into the corresponding Euler-Lagrange equation to yield

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{p}} \right] - \frac{\partial L}{\partial p} = e^{dt} (\ddot{p} + d\dot{p} + bp) = 0 \quad (5.7)$$

from which the equation of motion (5.4) can be extracted because  $e^{dt} \neq 0$  for all  $t$ .

### 5.1.2 Lagrangians for Classically Damped Systems

Suppose the coefficient matrices  $\mathbf{C}$  and  $\mathbf{K}$  are symmetric and positive definite in (5.1). Associated with equation (5.1) is the symmetric eigenvalue problem  $\mathbf{K}\mathbf{u} = \lambda\mathbf{u}$ . Owing to the positive definiteness of  $\mathbf{K}$ , all eigenvalues  $\lambda_j$  ( $j = 1, 2, \dots, n$ ) are positive, and the corresponding eigenvectors  $\mathbf{u}_j$  are real and orthonormal.

Define the modal matrix by  $\mathbf{U} = [\mathbf{u}_1 \ \dots \ \mathbf{u}_n]$ . If (5.1) is classically damped, then it is amenable to modal analysis. Using the modal transformation  $\mathbf{q} = \mathbf{U}\mathbf{p}$ , equation (5.1) becomes decoupled in the modal coordinate  $\mathbf{p} = [p_1 \ \dots \ p_n]^T$  and has a form as in (1.3) with

$$\mathbf{U}^T \mathbf{U} = \mathbf{I}, \quad \mathbf{D} = \mathbf{U}^T \mathbf{C} \mathbf{U} = \text{diag}[d_j], \quad \mathbf{\Omega} = \mathbf{U}^T \mathbf{K} \mathbf{U} = \text{diag}[b_j]. \quad (5.8)$$

Under the assumption of classical damping, Udwadia [29] decoupled (5.1) into  $n$  independent SDOF systems from which the Lagrangian functions

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} \left( \dot{\mathbf{q}}^T e^{\mathbf{C}t} \dot{\mathbf{q}} + \dot{\mathbf{q}}^T e^{\mathbf{C}t} \mathbf{C} \mathbf{q} + \frac{1}{2} \mathbf{q}^T e^{\mathbf{C}t} \mathbf{C}^2 \mathbf{q} \right) - \frac{1}{2} \mathbf{q}^T e^{\mathbf{C}t} \mathbf{K} \mathbf{q} \quad (5.9)$$

and

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} \dot{\mathbf{q}}^T e^{\mathbf{C}t} \dot{\mathbf{q}} - \frac{1}{2} \mathbf{q}^T e^{\mathbf{C}t} \mathbf{K} \mathbf{q}. \quad (5.10)$$

were constructed by using equations (5.5) and (5.6), respectively. Substitute either (5.9) or (5.10) into equation (5.3) to obtain

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{\mathbf{q}}} \right] - \frac{\partial L}{\partial \mathbf{q}} = e^{\mathbf{C}t} (\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q}) = \mathbf{0}, \quad (5.11)$$

from which the equations of motion (5.1) is recovered because  $\det(e^{\mathbf{C}t}) \neq 0$  for all  $t$ .

Practically speaking, classical damping implies that energy dissipation is almost uniformly distributed throughout a system. However, damping in linear systems is routinely non-classical [33].

## 5.2 Generalized Lagrangian Functions for Systems Possessing Distinct Eigenvalues

As explained in chapter 2, equation (5.1) can be decoupled into equation (1.3) using an extension of modal analysis. Upon decoupling, one obtains  $n$  independent SDOF systems of the form

$$\ddot{p}_j + d_j \dot{p}_j + b_j p_j = 0, \quad (5.12)$$

where  $d_j = -(\lambda_j + \lambda_{n+j})$  and  $b_j = \lambda_j \lambda_{n+j}$  ( $j = 1, 2, \dots, n$ ) are constants that populate the diagonal of the coefficient matrices  $\mathbf{D}$  and  $\mathbf{\Omega}$ , respectively. Recalling equation (5.5), a Lagrangian function associated with equation (5.12) is

$$L_j(p_j, \dot{p}_j, t) = \frac{1}{2} \left( \dot{p}_j^2 + d_j \dot{p}_j p_j + \frac{d_j^2}{2} p_j^2 \right) e^{d_j t} - \frac{b_j}{2} p_j^2 e^{d_j t}. \quad (5.13)$$

It follows that a Lagrangian function for the entire decoupled system (1.3) is given by [29]

$$\begin{aligned} L(\mathbf{p}, \dot{\mathbf{p}}, t) &= \sum_{j=1}^n L_j(p_j, \dot{p}_j, t) \\ &= \frac{1}{2} \left( \dot{\mathbf{p}}^T e^{\mathbf{D}t} \dot{\mathbf{p}} + \dot{\mathbf{p}}^T e^{\mathbf{D}t} \mathbf{D} \mathbf{p} + \frac{1}{2} \mathbf{p}^T e^{\mathbf{D}t} \mathbf{D}^2 \mathbf{p} \right) - \frac{1}{2} \mathbf{p}^T e^{\mathbf{D}t} \mathbf{\Omega} \mathbf{p}. \end{aligned} \quad (5.14)$$

It is straightforward to verify that equation (5.14) is indeed a Lagrangian function for the decoupled system (1.3) because the equation of motion is recovered from evaluating the associated Euler-Lagrange equation

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{\mathbf{p}}} \right] - \frac{\partial L}{\partial \mathbf{p}} = \mathbf{0}. \quad (5.15)$$

Using equation (2.26) (taking  $\mathbf{f} = \mathbf{0}$ ), the Lagrangian function  $L(\mathbf{p}, \dot{\mathbf{p}}, t)$  for the decoupled system can be expressed in terms of the original coordinate  $\mathbf{q}$ , resulting in a function

$$\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} \mathbf{q}^T \mathbf{A}_1(t) \mathbf{q} + \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{A}_2(t) \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{A}_3(t) \mathbf{q} \quad (5.16)$$

where

$$\mathbf{A}_1(t) = \mathbf{S}_3^T e^{\mathbf{D}t} \mathbf{S}_3 + \mathbf{S}_3^T e^{\mathbf{D}t} \mathbf{D} \mathbf{S}_1 + \frac{1}{2} \mathbf{S}_1^T e^{\mathbf{D}t} \mathbf{D}^2 \mathbf{S}_1 - \mathbf{S}_1^T e^{\mathbf{D}t} \mathbf{\Omega} \mathbf{S}_1, \quad (5.17)$$

$$\mathbf{A}_2(t) = \mathbf{S}_4^T e^{\mathbf{D}t} \mathbf{S}_4 + \mathbf{S}_4^T e^{\mathbf{D}t} \mathbf{D} \mathbf{S}_2 + \frac{1}{2} \mathbf{S}_2^T e^{\mathbf{D}t} \mathbf{D}^2 \mathbf{S}_2 - \mathbf{S}_2^T e^{\mathbf{D}t} \mathbf{\Omega} \mathbf{S}_2, \quad (5.18)$$

$$\mathbf{A}_3(t) = \mathbf{S}_4^T e^{\mathbf{D}t} \mathbf{S}_3 + \frac{1}{2} (\mathbf{S}_4^T e^{\mathbf{D}t} \mathbf{D} \mathbf{S}_1 + \mathbf{S}_2^T e^{\mathbf{D}t} \mathbf{D} \mathbf{S}_3 + \mathbf{S}_2^T e^{\mathbf{D}t} \mathbf{D}^2 \mathbf{S}_1) - \mathbf{S}_2^T e^{\mathbf{D}t} \mathbf{\Omega} \mathbf{S}_1. \quad (5.19)$$

Under the assumption of classical damping,  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  would be a Lagrangian function for the original system (5.1). This is precisely the approach adopted in [29] in the derivation



of equation (5.9). However,  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  as given by equation (5.16) generally does not satisfy the Euler-Lagrange equation in  $\mathbf{q}$ , i.e.,

$$\frac{d}{dt} \left[ \frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}} \right] - \frac{\partial \hat{L}}{\partial \mathbf{q}} \neq \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t) (\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q}) \quad (5.20)$$

for any  $\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t)$ . Thus,  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  is not a Lagrangian function for equation (5.1) even though, as a scalar function,  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  still provides compact storage of system properties. What equation is satisfied by  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ ? Can  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  generate the equation of motion?

### 5.2.1 Transformation of Euler-Lagrange Equations

The inverse of (2.26) is

$$\begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix}, \quad (5.21)$$

where each  $n \times n$  submatrix of

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_3 & \mathbf{T}_4 \end{bmatrix} \quad (5.22)$$

are given by equations (2.21)-(2.24). Denote the elements of  $\mathbf{T}_1$  and  $\mathbf{T}_3$  by  $T_{1,kj}$  and  $T_{3,kj}$  ( $k, j = 1, 2, \dots, n$ ), respectively. Using equation (5.21),

$$\begin{aligned} \frac{\partial L}{\partial p_j} &= \sum_{k=1}^n \left[ \frac{\partial \hat{L}}{\partial q_k} \frac{\partial q_k}{\partial p_j} + \frac{\partial \hat{L}}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial p_j} + \frac{\partial \hat{L}}{\partial t} \frac{\partial t}{\partial p_j} \right] \\ &= \sum_{k=1}^n \left[ \frac{\partial \hat{L}}{\partial q_k} T_{1,kj} + \frac{\partial \hat{L}}{\partial \dot{q}_k} T_{3,kj} \right]. \end{aligned} \quad (5.23)$$

As a consequence,

$$\frac{\partial L}{\partial \mathbf{p}} = \mathbf{T}_1^T \frac{\partial \hat{L}}{\partial \mathbf{q}} + \mathbf{T}_3^T \frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}}. \quad (5.24)$$

Likewise,

$$\frac{\partial L}{\partial \dot{\mathbf{p}}} = \mathbf{T}_2^T \frac{\partial \hat{L}}{\partial \mathbf{q}} + \mathbf{T}_4^T \frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}}. \quad (5.25)$$

Recall that  $L(\mathbf{p}, \dot{\mathbf{p}}, t)$  is a Lagrangian function for the decoupled system (1.3), satisfying equation (5.15). Substitute equations (5.24) and (5.25) into equation (5.15) to obtain

$$\left( \mathbf{T}_4^T \frac{d}{dt} \left[ \frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}} \right] - \mathbf{T}_1^T \frac{\partial \hat{L}}{\partial \mathbf{q}} \right) + \left( \mathbf{T}_2^T \frac{d}{dt} \left[ \frac{\partial \hat{L}}{\partial \mathbf{q}} \right] - \mathbf{T}_3^T \frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}} \right) = \mathbf{0}. \quad (5.26)$$

This is the equation satisfied by  $\hat{L}(\mathbf{p}, \dot{\mathbf{p}}, t)$ . Moreover, evaluation of this equation yields an equation from which the equation of motion (5.1) can be extracted. One would consider

$\hat{L}(\mathbf{p}, \dot{\mathbf{p}}, t)$  as a generalized Lagrangian function and equation (5.26) as a modified Euler-Lagrange equation.

Why does  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  satisfy the Euler-Lagrange equation (5.3) when system (5.1) is classically damped? Why is it necessary to use equation (5.26) in general to extract the equation of motion? If (1.1) represents a classically damped system, then equation (2.32) is applicable. In this case, the upper half of the decoupling transformation (2.26) reduces to the modal transformation  $\mathbf{q} = \mathbf{U} \mathbf{p}$  and the lower half reduces to  $\dot{\mathbf{q}} = \mathbf{U} \dot{\mathbf{p}}$ . The generalized Lagrangian function  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  in equation (5.16) reduces to the traditional Lagrangian function given by (5.9). Equivalently, under the assumption that system (5.1) is classically damped, (2.32) is applicable and equation (5.26) simplifies to a traditional Euler-Lagrange equation given by equation (5.3). Essentially, the state space decoupling transformation (2.26) or (5.21) becomes a configuration-space transformation under classical damping. A configuration space transformation modifies the Euler-Lagrange equation by introducing only a matrix multiplier, and essentially  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  satisfies equation (5.3). In general, equation (2.26) or equation (5.21) is a genuine state space transformation, which modifies the Euler-Lagrange equation to the form represented by equation (5.26).

In summary, system decoupling in real space, an approach utilized by Udwadia [29], always produces a scalar function  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ , which is either a Lagrangian function or a generalized Lagrangian function. In either case,  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  can be used to generate the equation of motion (5.1) and it contains information on system properties.

### 5.2.1.1 Illustrative Example: a Symmetric System

Consider a non-classically damped system specified by

$$\ddot{\mathbf{q}} + \begin{bmatrix} 0.7 & -0.1 \\ -0.1 & 0.2 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \mathbf{q} = \mathbf{0}. \quad (5.27)$$

Solution of the quadratic eigenvalue problem (2.8) yields

$$\mathbf{\Lambda}_1 = \begin{bmatrix} -0.1792 + 1.0008i & 0 \\ 0 & -0.2708 + 1.6819i \end{bmatrix}, \quad \mathbf{\Lambda}_2 = \overline{\mathbf{\Lambda}}_1 \quad (5.28)$$

and

$$\mathbf{V}_1 = \begin{bmatrix} 0.7328 - 0.0949i & 0.7152 + 0.1634i \\ 0.7180 + 0.0945i & -0.7118 + 0.1601i \end{bmatrix}, \quad \mathbf{V}_2 = \overline{\mathbf{V}}_1. \quad (5.29)$$

The real and diagonal coefficient matrices of the decoupled system (1.3) are given by

$$\mathbf{D} = \begin{bmatrix} 0.3584 & 0 \\ 0 & 0.5416 \end{bmatrix}, \quad \mathbf{\Omega} = \begin{bmatrix} 1.0337 & 0 \\ 0 & 2.9022 \end{bmatrix}. \quad (5.30)$$

The real decoupling transformations (2.26) and (5.21) are, respectively, defined by the matrices

$$\mathbf{S} = \begin{bmatrix} 0.6740 & 0.7294 & -0.0948 & 0.0944 \\ 0.7474 & -0.7282 & 0.0972 & 0.0952 \\ 0.2840 & -0.2836 & 0.7498 & 0.7011 \\ -0.0991 & -0.0932 & 0.6889 & -0.7376 \end{bmatrix}, \quad (5.31)$$

$$\mathbf{T} = \begin{bmatrix} 0.7158 & 0.7415 & -0.0948 & 0.0972 \\ 0.7349 & -0.6860 & 0.0944 & 0.0952 \\ 0.0980 & -0.2820 & 0.7498 & 0.6889 \\ -0.0976 & -0.2763 & 0.7011 & -0.7376 \end{bmatrix}. \quad (5.32)$$

The generalized Lagrangian function in equation (5.16) is given by

$$\begin{aligned} \hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = & e^{0.3584t} (-0.1455q_1^2 - 0.5543q_1q_2 - 0.2548q_2^2 + 0.2640\dot{q}_1^2 + 0.3607q_1\dot{q}_1 \\ & - 0.0427q_2\dot{q}_1 + 0.5351\dot{q}_2\dot{q}_1 + 0.2270q_1\dot{q}_2 - 0.1787q_2\dot{q}_2 + 0.2533\dot{q}_2^2) \\ & + e^{0.5416t} (-0.7847q_1^2 + 1.5096q_1q_2 - 0.7079q_2^2 + 0.2424\dot{q}_1^2 - 0.1316q_1\dot{q}_1 \\ & - 0.0075q_2\dot{q}_1 - 0.5352\dot{q}_2\dot{q}_1 - 0.2248q_1\dot{q}_2 + 0.4029q_2\dot{q}_2 + 0.2405\dot{q}_2^2). \end{aligned} \quad (5.33)$$

Using equations (5.31) and (5.32) to evaluate equation (5.26), one obtains

$$\begin{bmatrix} 0.7328 e^{0.3584t} & 0.7180 e^{0.3584t} \\ 0.7152 e^{0.5416t} & -0.7118 e^{0.5416t} \end{bmatrix} \times \left( \ddot{\mathbf{q}} + \begin{bmatrix} 0.7 & -0.1 \\ -0.1 & 0.2 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \mathbf{q} \right) = \mathbf{0}. \quad (5.34)$$

Observe that

$$\det \left( \begin{bmatrix} 0.7328 e^{0.3584t} & 0.7180 e^{0.3584t} \\ 0.7152 e^{0.5416t} & -0.7118 e^{0.5416t} \end{bmatrix} \right) = -1.0351 e^{0.9t} \neq 0 \quad (5.35)$$

for all  $t$ . Therefore, the equation of motion (5.27) can be extracted from equation (5.34). Indeed, the generalized Lagrangian function  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  generates the equation of motion specified by equation (5.27) from a modified Euler-Lagrange equation.

### 5.2.1.2 Illustrative Example: a Gyroscopic System

A gyroscopic system is defined by

$$\ddot{\mathbf{q}} + \begin{bmatrix} 0 & -0.2 \\ 0.2 & 0 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{q} = \mathbf{0}. \quad (5.36)$$

This is a realization of (5.1) with a non-symmetric coefficient matrix. Solution of the quadratic eigenvalue problem (2.8) yields

$$\mathbf{\Lambda}_1 = \begin{bmatrix} 0.9934i & 0 \\ 0 & 2.0132i \end{bmatrix}, \quad \mathbf{\Lambda}_2 = \overline{\mathbf{\Lambda}}_1 \quad (5.37)$$

and

$$\mathbf{V}_1 = \begin{bmatrix} -1.0022 & -0.1330i \\ 0.0661i & 1.0088 \end{bmatrix}, \quad \mathbf{V}_2 = \overline{\mathbf{V}}_1. \quad (5.38)$$

The real and diagonal coefficient matrices of the decoupled system (1.3) are given by

$$\mathbf{D} = \mathbf{0}, \quad \mathbf{\Omega} = \begin{bmatrix} 0.9869 & 0 \\ 0 & 4.0531 \end{bmatrix}. \quad (5.39)$$

The real decoupling transformations (2.26) and (5.21) are, respectively, defined by the matrices

$$\mathbf{S} = \begin{bmatrix} -0.9936 & 0 & 0 & -0.0651 \\ 0 & 0.9741 & 0.0647 & 0 \\ 0 & 0.2603 & -0.9805 & 0 \\ -0.0647 & 0 & 0 & 0.9870 \end{bmatrix}, \quad (5.40)$$

$$\mathbf{T} = \begin{bmatrix} -1.0022 & 0 & 0 & -0.0661 \\ 0 & 1.0088 & 0.0665 & 0 \\ 0 & 0.2678 & -1.0022 & 0 \\ -0.0657 & 0 & 0 & 1.0088 \end{bmatrix}. \quad (5.41)$$

The generalized Lagrangian function in equation (5.16) is

$$\begin{aligned} \hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = & -0.4850q_1^2 - 1.8890q_2^2 - 0.5106q_2\dot{q}_1 + 0.4723\dot{q}_1^2 \\ & - 0.1276q_1\dot{q}_2 + 0.4850q_2^2. \end{aligned} \quad (5.42)$$

Using equations (5.40) and (5.41) to evaluate equation (5.26), one obtains

$$\begin{bmatrix} -0.9805 & 0 \\ 0 & 0.9870 \end{bmatrix} \times \left( \ddot{\mathbf{q}} + \begin{bmatrix} 0 & -0.2 \\ 0.2 & 0 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{q} \right) = \mathbf{0}. \quad (5.43)$$

It follows that the equation of motion (5.36) can be extracted. This example demonstrates that systems with non-symmetric coefficients can be readily treated.

### 5.2.2 Relating Lagrangians for Coupled and Decoupled Systems

It was demonstrated that it is always possible to obtain a Lagrangian  $L(\mathbf{p}, \dot{\mathbf{p}}, t)$  for the decoupled system (1.3) because one can use phase synchronization to transform equation (5.1) into equation (1.3). However, the change of variables from  $\mathbf{p}$  to  $\mathbf{q}$ , given by equation (2.26), results in a scalar function  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  that is generally not a Lagrangian in  $\mathbf{q}$  because it satisfies a modified version of the Euler-Lagrange equation (5.26). Now, the following question is addressed: when will decoupling yield a Lagrangian in  $\mathbf{q}$  from any Lagrangian  $L(\mathbf{p}, \dot{\mathbf{p}}, t)$ ?

In a study of this topic, Udwadia showed that it is always possible to obtain a Lagrangian for system (5.1) when its coefficients  $\mathbf{C}$  and  $\mathbf{K}$  are simultaneously diagonalizable by a similarity transformation in the  $n$ -dimensional configuration space [30], and thus they must

commute in multiplication:  $\mathbf{CK} = \mathbf{KC}$ . This result pertains not only to classically damped linear systems [29] (i.e., systems with symmetric, positive definite, and diagonalizable  $\mathbf{C}$  and  $\mathbf{K}$  that commute [18]) but to systems with non-symmetric coefficients as well [30].

It will be shown that, starting from any Lagrangian for the decoupled system (1.3) generated via phase synchronization, a corresponding Lagrangian for the coupled system (5.1) can be obtained if and only if the coefficients  $\mathbf{C}$  and  $\mathbf{K}$  (that need not be symmetric or positive definite) are simultaneously diagonalizable by a similarity transformation in the configuration space. Essentially, the findings in [30] are verified, but with a stronger argument: it will be shown that simultaneous diagonalization of  $\mathbf{C}$  and  $\mathbf{K}$  is both necessary and sufficient to determine a Lagrangian in  $\mathbf{q}$  from any Lagrangian in  $\mathbf{p}$  while in [30] only a sufficient condition is provided.

**Theorem 5.1.** *Let the quadratic eigenvalue problem (2.8) associated with system (5.1) admit only simple eigenvalues. The function  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  is a Lagrangian in  $\mathbf{q}$  for system (5.1) if and only if  $\mathbf{C}$  and  $\mathbf{K}$  are simultaneously diagonalizable by a similarity transformation in the configuration space.*

*Proof.* To establish sufficiency of Theorem 5.1, assume  $\mathbf{C}$  and  $\mathbf{K}$  are simultaneously diagonalizable such that  $\mathbf{D} = \mathbf{V}^{-1}\mathbf{CV}$  and  $\mathbf{\Omega} = \mathbf{V}^{-1}\mathbf{KV}$ , where  $\mathbf{V} = [V_{sj}]$  ( $s, j = 1, 2, \dots, n$ ) is a real, constant and invertible  $n \times n$  matrix [31]. This implies that (5.1) is decoupled into (1.3) through a real configuration space transformation  $\mathbf{q} = \mathbf{V}\mathbf{p}$  by first applying this transformation to (5.1) and then multiplying the resulting expression by  $\mathbf{V}^{-1}$  on the left. Let  $L(\mathbf{p}, \dot{\mathbf{p}}, t)$  be any Lagrangian in  $\mathbf{p}$ . Note that

$$\frac{\partial L}{\partial p_j} = \sum_{s=1}^n \left[ \frac{\partial \hat{L}}{\partial q_s} \frac{\partial q_s}{\partial p_j} + \frac{\partial \hat{L}}{\partial \dot{q}_s} \frac{\partial \dot{q}_s}{\partial p_j} + \frac{\partial \hat{L}}{\partial t} \frac{\partial t}{\partial p_j} \right] = \sum_{s=1}^n \frac{\partial \hat{L}}{\partial q_s} V_{sj}, \quad (5.44)$$

where  $L(\mathbf{p}, \dot{\mathbf{p}}, t) = L(\mathbf{V}^{-1}\mathbf{q}, \mathbf{V}^{-1}\dot{\mathbf{q}}, t) = \hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ , and

$$\frac{\partial L}{\partial \mathbf{p}} = \mathbf{V}^T \frac{\partial \hat{L}}{\partial \mathbf{q}}. \quad (5.45)$$

Likewise,

$$\frac{\partial L}{\partial \dot{\mathbf{p}}} = \mathbf{V}^T \frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}}. \quad (5.46)$$

Inserting (5.45) and (5.46) into the decoupled system Euler-Lagrange equation (5.15) yields

$$\mathbf{V}^T \left\{ \frac{d}{dt} \left[ \frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}} \right] - \frac{\partial \hat{L}}{\partial \mathbf{q}} \right\} = \mathbf{0}, \quad (5.47)$$

from which we can extract the Euler-Lagrange equation in  $\mathbf{q}$  because  $\mathbf{V}^T$  is nonsingular. Thus, the function  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  is a Lagrangian in  $\mathbf{q}$ , concluding the proof of sufficiency.

To demonstrate necessity, suppose any Lagrangian  $L(\mathbf{p}, \dot{\mathbf{p}}, t)$  for the decoupled system (1.3), obtained via phase synchronization, is converted into a function  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  by applying the coordinate transformation (2.26). If  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  is a Lagrangian in  $\mathbf{q}$ , then it must satisfy the corresponding Euler-Lagrange equation up to a real, nonsingular,  $n \times n$  matrix multiplier  $\mathbf{F}$ :

$$\mathbf{F} \left( \frac{d}{dt} \left[ \frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}} \right] - \frac{\partial \hat{L}}{\partial \mathbf{q}} \right) = \mathbf{0}. \quad (5.48)$$

When the eigenvalues of (2.8) are distinct, equation (5.48) is satisfied for all  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  so long as

$$\mathbf{T}_1 = \mathbf{T}_4 \neq \mathbf{0}, \quad (5.49)$$

and

$$\mathbf{T}_2^T \frac{d}{dt} \left[ \frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}} \right] - \mathbf{T}_3^T \frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}} = \mathbf{0}. \quad (5.50)$$

To verify this, it is first necessary to confirm that (5.49) implies (5.50). From the definitions of  $\mathbf{T}_1$  and  $\mathbf{T}_4$  in equations (2.21) and (2.24), respectively,  $\mathbf{T}_1 = \mathbf{T}_4 \neq \mathbf{0}$  implies  $\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}$  because all eigenvalues are distinct. This results in  $\mathbf{T}_1 = \mathbf{T}_4 = \mathbf{V}$  and in  $\mathbf{T}_2 = \mathbf{T}_3 = \mathbf{0}$  from equations (2.22) and (2.23), and thus condition (5.50) is satisfied for any  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ . Now it is necessary to check that equation (5.50) implies equation (5.49). Here, there are multiple cases to consider:

1.  $\mathbf{T}_2 = \mathbf{T}_3 = \mathbf{0}$ : This is possible according to equations (2.22) and (2.23) only when  $\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}$  because the eigenvalues are simple. This in turn leads to  $\mathbf{T}_1 = \mathbf{T}_4 = \mathbf{V}$ , and hence condition (5.49) is satisfied.
2.  $\mathbf{T}_2 \neq \mathbf{0}$  and  $\mathbf{T}_3 \neq \mathbf{0}$ : if this is true, then  $\mathbf{V}_1 \neq \mathbf{V}_2$ , which means  $\mathbf{T}_1 = \mathbf{T}_4$  can never be attained, contradicting (5.49).
3.  $\mathbf{T}_2 \neq \mathbf{0}$  and  $\mathbf{T}_3 = \mathbf{0}$ : the definitions of  $\mathbf{T}_2$  and  $\mathbf{T}_3$  prohibit this.
4.  $\mathbf{T}_2 = \mathbf{0}$  and  $\mathbf{T}_3 \neq \mathbf{0}$ : again, this is not possible based on the definitions of  $\mathbf{T}_2$  and  $\mathbf{T}_3$ .

Therefore, we must have  $\mathbf{T}_2 = \mathbf{T}_3 = \mathbf{0}$  and  $\mathbf{T}_1 = \mathbf{T}_4 = \mathbf{V}$ . Consequently, equation (5.26) reduces to equation (5.48), for which the matrix multiplier  $\mathbf{F} = \mathbf{T}_1^T = \mathbf{T}_4^T = \mathbf{V}^T$  is a constant. Also, the decoupling transformation (5.21) simplifies to the real configuration space mapping  $\mathbf{q} = \mathbf{V}\mathbf{p}$ . Applying the inverse of this transformation to the decoupled system (1.3) and multiplying the result on the left by  $\mathbf{V}$  produces (5.1) with  $\mathbf{C} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$  and  $\mathbf{K} = \mathbf{V}\mathbf{\Omega}\mathbf{V}^{-1}$ . Equivalently,  $\mathbf{D} = \mathbf{V}^{-1}\mathbf{C}\mathbf{V}$  and  $\mathbf{\Omega} = \mathbf{V}^{-1}\mathbf{K}\mathbf{V}$ , and hence  $\mathbf{C}$  and  $\mathbf{K}$  are simultaneously diagonalizable by a configuration space similarity transformation. This concludes the proof of necessity. ■

### 5.2.2.1 Illustrative Example: Simultaneously Diagonalizable Coefficients

The linear system

$$\ddot{\mathbf{q}} + \begin{bmatrix} 1 & 3 \\ -1 & 9 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 16 & -6 \\ 2 & 0 \end{bmatrix} \mathbf{q} = \mathbf{0} \quad (5.51)$$

is such that

$$\mathbf{C}\mathbf{K} = \mathbf{K}\mathbf{C} = \begin{bmatrix} 22 & -6 \\ 2 & 6 \end{bmatrix}. \quad (5.52)$$

It can be checked that this system is diagonalized in configuration space by the matrix

$$\mathbf{V} = \begin{bmatrix} 4 + \sqrt{13} & 4 - \sqrt{13} \\ 1 & 1 \end{bmatrix}. \quad (5.53)$$

The real and diagonal coefficient matrices of the decoupled system (1.3) are given by

$$\mathbf{D} = \begin{bmatrix} 5 - \sqrt{13} & 0 \\ 0 & 5 + \sqrt{13} \end{bmatrix}, \quad \mathbf{\Omega} = \begin{bmatrix} 8 + 2\sqrt{13} & 0 \\ 0 & 8 - 2\sqrt{13} \end{bmatrix}. \quad (5.54)$$

Decoupling by similarity transformation is simply phase synchronization with  $\mathbf{T}_2 = \mathbf{T}_3 = \mathbf{0}$ ,  $\mathbf{T}_1 = \mathbf{T}_4 = \mathbf{V}$ ,  $\mathbf{S}_2 = \mathbf{S}_3 = \mathbf{0}$  and  $\mathbf{S}_1 = \mathbf{S}_4 = \mathbf{V}^{-1}$ . A Lagrangian for the decoupled system is

$$\begin{aligned} L(\mathbf{p}, \dot{\mathbf{p}}, t) &= \frac{1}{2} \left[ -(8 + 2\sqrt{13})p_1^2 + \dot{p}_1^2 \right] e^{(5-\sqrt{13})t} \\ &\quad + \frac{1}{2} \left[ -(8 - 2\sqrt{13})p_2^2 + \dot{p}_2^2 \right] e^{(5+\sqrt{13})t}. \end{aligned} \quad (5.55)$$

It can be readily checked that

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{\mathbf{p}}} \right] - \frac{\partial L}{\partial \mathbf{p}} = \begin{bmatrix} e^{(5-\sqrt{13})t} & 0 \\ 0 & e^{(5+\sqrt{13})t} \end{bmatrix} \times (\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \mathbf{\Omega}\mathbf{p}) = \mathbf{0}. \quad (5.56)$$

Changing variables back to  $\mathbf{q}$  and  $\dot{\mathbf{q}}$ :

$$\begin{aligned} \hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t) &= \frac{e^{(5-\sqrt{13})t}}{104} [\dot{q}_1^2 - (8 - 2\sqrt{13})\dot{q}_1\dot{q}_2 + (29 - 8\sqrt{13})\dot{q}_2^2 \\ &\quad - (8 + 2\sqrt{13})q_1^2 + 12q_1q_2 - (24 - 6\sqrt{13})q_2^2] \\ &\quad + \frac{e^{(5+\sqrt{13})t}}{104} [\dot{q}_1^2 - (8 + 2\sqrt{13})\dot{q}_1\dot{q}_2 + (29 + 8\sqrt{13})\dot{q}_2^2 \\ &\quad - (8 - 2\sqrt{13})q_1^2 + 12q_1q_2 - (24 + 6\sqrt{13})q_2^2]. \end{aligned} \quad (5.57)$$

The function  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  satisfies

$$\frac{d}{dt} \left[ \frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}} \right] - \frac{\partial \hat{L}}{\partial \mathbf{q}} = \mathbf{F}(\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q}) = \mathbf{0}, \quad (5.58)$$

where the invertible matrix multiplier is given by

$$\mathbf{F} = \frac{1}{52} \begin{bmatrix} e^{(5-\sqrt{13})t} + e^{(5+\sqrt{13})t} & -(4-\sqrt{13})e^{(5-\sqrt{13})t} - (4+\sqrt{13})e^{(5+\sqrt{13})t} \\ \text{Sym.} & (29-8\sqrt{13})e^{(5-\sqrt{13})t} + (29+8\sqrt{13})e^{(5+\sqrt{13})t} \end{bmatrix}. \quad (5.59)$$

Thus,  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  is a Lagrangian in  $\mathbf{q}$ .

### 5.3 Generalized Lagrangian Functions for Defective Systems

Lagrangian function  $L(\mathbf{p}, \dot{\mathbf{p}}, t)$  for the decoupled system (1.3), whether or not equation (5.1) is defective, can always be expressed as in equation (5.14). When  $L(\mathbf{p}, \dot{\mathbf{p}}, t)$  is expressed in terms of the original system coordinate  $\mathbf{q}$ , the resulting function  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  still has the form given by equation (5.16), but the submatrices  $\mathbf{S}_k$  may be time-varying. Because  $\mathbf{T}_k$  may also be time-varying for defective systems, it can be shown that the modified Euler-Lagrange equation satisfied by  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  has the form

$$\left\{ \mathbf{T}_4^T \frac{d}{dt} \left[ \frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}} \right] - \left( \mathbf{T}_1^T - \dot{\mathbf{T}}_2^T \right) \frac{\partial \hat{L}}{\partial \mathbf{q}} \right\} + \left\{ \mathbf{T}_2^T \frac{d}{dt} \left[ \frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}} \right] - \left( \mathbf{T}_3^T - \dot{\mathbf{T}}_4^T \right) \frac{\partial \hat{L}}{\partial \mathbf{q}} \right\} = \mathbf{0}. \quad (5.60)$$

This is a generalization of equation (5.26) when system (5.1) is defective. As in the nondefective case, equation (5.60) implies that  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$  is generally not a Lagrangian function for the defective system (5.1), but evaluation of (5.60) allows the equation of motion (5.1) to be unpacked from  $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ .

#### 5.3.1 Illustrative Example: a Defective System

Consider a non-classically damped system of the form (5.1) specified by

$$\ddot{\mathbf{q}} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} \mathbf{q} = \mathbf{0}, \quad (5.61)$$

Solution of the quadratic eigenvalue problem (2.8) indicates that the system is defective with a repeated complex eigenvalue such that

$$\mathbf{J} = \begin{bmatrix} -1 + i\sqrt{2} & 1 \\ 0 & -1 + i\sqrt{2} \end{bmatrix}, \quad (5.62)$$

$$\mathbf{V} = \begin{bmatrix} -i\sqrt{2} & 3 \\ 1 & 0 \end{bmatrix}, \quad (5.63)$$

$$\mathbf{\Lambda} = (-1 + i\sqrt{2})\mathbf{I} \quad (5.64)$$



and

$$\mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad (5.65)$$

The real and diagonal coefficient matrices of the decoupled system (1.3) are given by

$$\mathbf{D} = 2\mathbf{I}, \quad \mathbf{\Omega} = 3\mathbf{I}. \quad (5.66)$$

The real decoupling transformations (2.69) and (2.62) are, respectively, defined by the matrices

$$\mathbf{S} = \begin{bmatrix} -t/4 & 1-t/4 & 0 & -t/4 \\ 1/4 & 1/4 & 0 & 1/4 \\ -t/4-1/4 & 5t/4-1/4 & -t/2 & t/4+3/4 \\ 1/4 & -5/4 & 1/2 & -1/4 \end{bmatrix}, \quad (5.67)$$

$$\mathbf{T} = \begin{bmatrix} -1 & 3-t & -1 & -t \\ 1 & t & 0 & 0 \\ 3 & 3t-1 & 1 & t+2 \\ 0 & 1 & 1 & t \end{bmatrix}. \quad (5.68)$$

The generalized Lagrangian function in equation (5.16) is

$$\begin{aligned} \hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = & e^{2t} \left( \frac{3}{32} \dot{q}_1^2 - \frac{9}{32} \dot{q}_2^2 + \frac{1}{8} \dot{q}_1^2 + \frac{9}{32} \dot{q}_2^2 - \frac{13}{16} q_1 \dot{q}_2 + \frac{1}{4} q_1 \dot{q}_1 - \frac{5}{16} q_1 \dot{q}_2 - \frac{1}{2} q_2 \dot{q}_1 + \frac{7}{16} q_2 \dot{q}_2 \right) \\ & + te^{2t} \left( \frac{1}{8} \dot{q}_1^2 + \frac{5}{4} \dot{q}_2^2 - \frac{1}{8} q_1 \dot{q}_2 + \frac{1}{8} q_1 \dot{q}_1 - \frac{3}{8} q_1 \dot{q}_2 - \frac{3}{8} q_2 \dot{q}_1 + \frac{5}{4} q_2 \dot{q}_2 - \frac{3}{8} \dot{q}_2 \dot{q}_2 \right) \\ & + t^2 e^{2t} \left( \frac{1}{16} \dot{q}_1^2 + \frac{7}{16} \dot{q}_2^2 + \frac{1}{8} \dot{q}_1^2 - \frac{1}{16} \dot{q}_2^2 - \frac{5}{8} q_1 \dot{q}_2 + \frac{1}{4} q_1 \dot{q}_1 - \frac{1}{8} q_1 \dot{q}_2 - \frac{1}{2} q_2 \dot{q}_1 - \frac{1}{8} q_2 \dot{q}_2 \right). \end{aligned} \quad (5.69)$$

Using equations (5.67) and (5.68) to evaluate equation (5.60), one obtains

$$\begin{bmatrix} -te^{2t}/2 & 3e^{2t}/4 \\ e^{2t}/2 & 0 \end{bmatrix} \times \left( \ddot{\mathbf{q}} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} \mathbf{q} \right) = \mathbf{0}. \quad (5.70)$$

Since the matrix multiplier is nonsingular for all  $t$ , the equation of motion (5.61) is extracted from equation (5.70). This example demonstrates that defective systems can indeed be tackled.

## 5.4 Existence of Lagrangians for Linear Systems

In this section, it will be shown that some coupled linear systems do not admit Lagrangian functions. Most of the Lagrangian functions for MDOF linear systems reported in the literature [29, 30, 54, 56, 59] are bilinear of the form

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} \mathbf{q}^T \mathbf{A}_1(t) \mathbf{q} + \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{A}_2(t) \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{A}_3(t) \mathbf{q}, \quad (5.71)$$

where  $\mathbf{A}_j(t)$  ( $j = 1, 2, 3$ ) are real  $n \times n$  matrices that may depend on time. Note that equation (5.71) is a general form that accommodates both standard and nonstandard Lagrangians. In addition, suppose that, for example, it is required that the kinetic energy of equation (5.1) be expressible as a quadratic form of the velocities and the coefficients of the quadratic form can be time-varying. Then a Lagrangian function for this subclass of equation (5.1), if it exists, is reducible to the form given by equation (5.71). Evaluating the Euler-Lagrange equation (5.3) with equation (5.71) and matching coefficients yields the system of equations

$$\frac{1}{2} [\mathbf{A}_2(t) + \mathbf{A}_2^T(t)] = \mathbf{Y}(t), \quad (5.72)$$

$$\frac{1}{2} [\dot{\mathbf{A}}_2(t) + \dot{\mathbf{A}}_2^T(t)] + [\mathbf{A}_3(t) - \mathbf{A}_3^T(t)] = \mathbf{Y}(t)\mathbf{C}, \quad (5.73)$$

$$\dot{\mathbf{A}}_3^T(t) - \frac{1}{2} [\mathbf{A}_1(t) + \mathbf{A}_1^T(t)] = \mathbf{Y}(t)\mathbf{K}. \quad (5.74)$$

These equations imply that  $\mathbf{Y}(t) \neq \mathbf{0}$  is symmetric and satisfies the matrix differential equation

$$\ddot{\mathbf{Y}}(t) - \dot{\mathbf{Y}}(t)\mathbf{C} = \mathbf{K}^T\mathbf{Y}(t) - \mathbf{Y}(t)\mathbf{K}. \quad (5.75)$$

However, due to the symmetry of  $\mathbf{Y}(t)$ , equation (5.75) constitutes an overdetermined system of differential equations. There are  $n^2$  scalar differential equations associated with equation (5.75), but only  $n(n+1)/2$  solutions are needed. With  $\mathbf{C}$  and  $\mathbf{K}$  arbitrary for system (5.1), it is generally not possible to make the  $n^2$  scalar differential equations consistent, so an admissible nontrivial solution  $\mathbf{Y}(t)$  does not exist. Thus, system (5.1) will generally not admit Lagrangian functions of the form given by equation (5.71) unless restrictions are placed on the coefficient matrices  $\mathbf{C}$  and  $\mathbf{K}$ . It is important to note that the existence of an acceptable solution  $\mathbf{Y}(t)$  does not guarantee the existence of a corresponding Lagrangian function. However, if there is no admissible solution to (5.75), then a Lagrangian does not exist for system (5.1).

To examine equation (5.75) more intimately, assume that (5.1) has two degrees of freedom, with

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \quad \mathbf{Y}(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_2(t) & y_3(t) \end{bmatrix}. \quad (5.76)$$

In this case, the four component equations associated with equation (5.75) can be written explicitly as

$$\ddot{y}_1 - C_{11}\dot{y}_1 - C_{21}\dot{y}_2 = 0, \quad (5.77)$$

$$\ddot{y}_2 - C_{12}\dot{y}_1 - C_{22}\dot{y}_2 + K_{12}y_1 + (K_{22} - K_{11})y_2 - K_{21}y_3 = 0, \quad (5.78)$$

$$\ddot{y}_2 - C_{11}\dot{y}_2 - C_{21}\dot{y}_3 - K_{12}y_1 - (K_{22} - K_{11})y_2 + K_{21}y_3 = 0, \quad (5.79)$$

$$\ddot{y}_3 - C_{12}\dot{y}_2 - C_{22}\dot{y}_3 = 0. \quad (5.80)$$

Notice that  $y_2$  must satisfy simultaneously two differential equations: equations (5.78) and (5.79). Subtract equation (5.79) from (5.78) to obtain

$$-C_{12}\dot{y}_1 + (C_{11} - C_{22})\dot{y}_2 + C_{21}\dot{y}_3 + 2K_{12}y_1 + 2(K_{22} - K_{11})y_2 - 2K_{21}y_3 = 0. \quad (5.81)$$

If a Lagrangian function of the form (5.71) exists, there must be at least one nontrivial solution to equation (5.81) for which  $y_j$  ( $j = 1, 2, 3$ ) are not all zero. Since the elements  $C_{js}$  and  $K_{js}$  ( $j, s = 1, 2$ ) are arbitrary, the only way equation (5.81) will always be satisfied is for  $y_j = 0$  and  $\dot{y}_j = 0$ , which contradicts the requirement that some  $y_j$  be nontrivial. Consequently,  $L(\mathbf{q}, \dot{\mathbf{q}}, t)$  defined by equation (5.71) is a Lagrangian function for a subclass of systems (5.1) only, i.e., there are systems of the form given by (5.1) that do not admit Lagrangian functions.

### 5.4.1 Deductions Using Helmholtz Conditions

If system (5.1) possesses a Lagrangian function  $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ , the corresponding Euler-Lagrange equation (5.3) generates the system of equations

$$\mathbf{G}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t) = \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t)(\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q}). \quad (5.82)$$

Equation (5.82) must satisfy Helmholtz conditions, which are necessary and sufficient conditions for the existence of  $L(\mathbf{q}, \dot{\mathbf{q}}, t)$  and can be specified in component form as [55]

$$\begin{aligned} \frac{\partial G_s}{\partial \ddot{q}_j} &= \frac{\partial G_j}{\partial \ddot{q}_s}, \\ \frac{\partial G_s}{\partial \dot{q}_j} - \frac{\partial G_j}{\partial \dot{q}_s} &= \frac{1}{2} \frac{d}{dt} \left[ \frac{\partial G_s}{\partial \dot{q}_j} - \frac{\partial G_j}{\partial \dot{q}_s} \right], \\ \frac{\partial G_s}{\partial \dot{q}_j} - \frac{\partial G_j}{\partial \dot{q}_s} &= \frac{d}{dt} \left[ \frac{\partial G_s}{\partial \ddot{q}_j} - \frac{\partial G_j}{\partial \ddot{q}_s} \right], \end{aligned} \quad (5.83)$$

where  $s, j = 1, 2, \dots, n$ . Consider a subclass of equation (5.1) with  $\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t) = \mathbf{Y}(t)$ . The  $j$ th component of equation (5.82) is given by

$$G_j(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t) = \sum_{s=1}^n Y_{js} \ddot{q}_s + \sum_{m,s=1}^n Y_{jm} C_{ms} \dot{q}_s + \sum_{m,s=1}^n Y_{jm} K_{ms} q_s \quad (5.84)$$

where  $j = 1, 2, \dots, n$  and  $\mathbf{Y} = [Y_{sj}]$ ,  $\mathbf{C} = [C_{sj}]$ , and  $\mathbf{K} = [K_{sj}]$ . Note that

$$\frac{\partial G_s}{\partial \ddot{q}_j} = Y_{sj}, \quad \frac{\partial G_s}{\partial \dot{q}_j} = \sum_{m=1}^n Y_{sm} C_{mj}, \quad \frac{\partial G_s}{\partial q_j} = \sum_{m=1}^n Y_{sm} K_{mj}. \quad (5.85)$$

The first part of equation (5.83) implies that  $Y_{sj} = Y_{js}$ , and thus  $\mathbf{Y}(t)$  must be symmetric. The second part of equation (5.83) yields

$$\sum_{m=1}^n Y_{sm} K_{mj} - \sum_{m=1}^n Y_{jm} K_{ms} = \frac{1}{2} \sum_{m=1}^n [\dot{Y}_{sm} C_{mj} - \dot{Y}_{jm} C_{ms}] \quad (5.86)$$

and therefore

$$\mathbf{Y}\mathbf{K} - \mathbf{K}^T\mathbf{Y} = \frac{1}{2} [\dot{\mathbf{Y}}\mathbf{C} - \mathbf{C}^T\dot{\mathbf{Y}}]. \quad (5.87)$$

Likewise, the third part of equation (5.83) gives

$$\mathbf{C}^T\mathbf{Y} = 2\dot{\mathbf{Y}}\mathbf{C} - \mathbf{Y}\mathbf{C}. \quad (5.88)$$

Differentiating equation (5.88) and combining with equation (5.87) results in (5.75). Thus, application of the Helmholtz conditions leads to the same conclusion regarding the existence of Lagrangian functions as before.

### 5.4.2 Illustrative Example: Construction of Lagrangian Function

Suppose system (5.1) is gyroscopic with skew-symmetric  $\mathbf{C}$  and symmetric and positive definite  $\mathbf{K}$ . Substitute  $C_{21} = -C_{12}$  and  $K_{21} = K_{12}$  into equation (5.81) to obtain

$$-C_{12}(\dot{y}_1 + \dot{y}_3) + (C_{11} - C_{22})\dot{y}_2 + 2K_{12}(y_1 - y_3) + 2(K_{22} - K_{11})y_2 = 0. \quad (5.89)$$

The remaining elements of  $\mathbf{C}$  and  $\mathbf{K}$  are arbitrary, and thus equation (5.89) is satisfied only when

$$y_1 = y_3 = \delta(t), \quad y_2 = 0, \quad \dot{y}_2 = 0, \quad \dot{y}_1 = -\dot{y}_3. \quad (5.90)$$

However,  $y_1 = y_3 = \delta(t)$  and  $\dot{y}_1 = -\dot{y}_3$  cannot be simultaneously satisfied unless  $\delta(t) = 0$ , which is not admissible because this results in a trivial solution  $\mathbf{Y}(t) = \mathbf{0}$ . Therefore, it must be that  $\delta(t) = \delta = \text{constant} \neq 0$ , so an admissible solution to equation (5.75) is  $\mathbf{Y}(t) = \delta\mathbf{I}$ , which can be easily verified by direct substitution.

Does a Lagrangian exist if  $\mathbf{Y}(t) = \delta\mathbf{I}$ ? Let  $\delta = 1$  so  $\mathbf{Y}(t) = \mathbf{I}$ . Equation (5.72) implies that  $\mathbf{A}_2(t) = \mathbf{I} + \mathbf{R}_1(t)$ , where  $\mathbf{R}_1(t)$  is a skew-symmetric matrix. Choose  $\mathbf{R}_1(t) = \mathbf{0}$  for simplicity. In this case, equation (5.73) reduces to  $\mathbf{A}_3(t) - \mathbf{A}_3^T(t) = \mathbf{C}$ . But  $\mathbf{C}$  is skew-symmetric, and therefore  $\mathbf{A}_3(t) = \mathbf{C}/2 + \mathbf{R}_2(t)$ , where  $\mathbf{R}_2(t)$  is a symmetric matrix. Choose  $\mathbf{R}_2(t) = \mathbf{0}$  for convenience. It follows from equation (5.74) that  $\mathbf{A}_1(t) = -\mathbf{K} + \mathbf{R}_3(t)$ , where  $\mathbf{R}_3(t)$  is skew-symmetric. Let  $\mathbf{R}_3(t) = \mathbf{0}$  for convenience. Then  $\mathbf{A}_1(t) = -\mathbf{K}$ ,  $\mathbf{A}_2(t) = \mathbf{I}$  and  $\mathbf{A}_3(t) = \mathbf{C}/2$ . Thus, the gyroscopic system (5.1) admits the Lagrangian function

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T\dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q} + \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{K}\mathbf{q}, \quad (5.91)$$

which was reported by Udwadia and Cho [56].

## 5.5 Conclusions

A comprehensive study of the evaluation of Lagrangian functions for linear systems has been reported. The major results, summarized in the following statements, are applicable to both non-defective and defective linear systems possessing either symmetric or non-symmetric coefficient matrices.

1. While Lagrangian functions for decoupled linear systems can be readily found, coupled linear systems may or may not admit Lagrangian functions.
2. Using an extension of modal analysis, any linear system can be decoupled in real space. Subsequently, a scalar function that plays the role of a Lagrangian function can be determined. This scalar function is either a traditional Lagrangian function or a generalized Lagrangian function.
3. A generalized Lagrangian function determined by system decoupling still produces the equation of motion and it still contains information on system properties. However, it satisfies a modified version of the Euler-Lagrange equation.

Given that many coupled linear systems do not admit traditional Lagrangian functions, generalized Lagrangian functions may be the best that one can achieve. Subject to this interpretation, a solution to the inverse problem of linear Lagrangian dynamics has been provided. It was also demonstrated that generalized Lagrangians coincide with Lagrangians when the coefficient matrices can be simultaneously diagonalized by a similarity transformation in configuration space.

# Chapter 6

## Conclusion

The three problems in the dynamics of linear systems studied in this dissertation have once been impeded by coordinate coupling, i.e., partial solution to these problems required explicitly, or implicitly, coordinate decoupling, such as modal analysis and its underlying assumptions (e.g., classical damping). Using earlier works [23–27] on the decoupling of linear systems, the method of phase synchronization was used as the main theoretical tool to tackle these problems. The main conclusions are summarized in the following statements:

1. It has been shown that almost all linear systems governed by equation (1.1) can be reduced to a canonical form specified by equation (3.1), a decoupled equation devoid of the velocity term and with the identity matrix as the coefficient of acceleration. The canonical form specified by equation (3.1) is the simplest representation of linear systems and all parameters required to construct the invertible transformation to convert equation (1.1) into equation (3.1) are obtained through the solution of the quadratic eigenvalue problem (2.8). As an important by-product, a solution to the well-trodden problem of reducing a damped passive system to an undamped form has been provided.
2. Using the methodology of phase synchronization, two methods were developed to determine oscillatory behavior of MDOF damped systems in free motion. It was shown how the system's spectrum can be used to determine oscillatory behavior: complex conjugate eigenvalues form underdamped degrees of freedom, while real eigenvalues generate overdamped or critically damped degrees of freedom. Further, a damping ratio for MDOF systems was constructed in (4.30). This damping ratio is a direct extension of the damping ratio for SDOF systems in (4.5) and was shown to be invariant under any linear transformation, including modal transformations. The damping ratio for systems (4.30) predicts oscillatory behavior whenever  $0 < \zeta < 1$ .
3. Lastly, a comprehensive study was reported herein for the evaluation of Lagrangian functions for linear systems possessing symmetric or non-symmetric coefficient matrices. Using phase synchronization, a scalar function that plays the role of a Lagrangian function can be determined. This scalar function is either a traditional Lagrangian

function or a generalized Lagrangian function. A generalized Lagrangian function determined by system decoupling still produces the equation of motion and it still contains information on system properties. However, it satisfies a modified version of the Euler-Lagrange equation. It was shown that generalized Lagrangians are in fact traditional Lagrangians whenever system (1.1) can be decoupled into (1.3) by a configuration space similarity transformation.

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